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Finite-dimensional adaptive observer design for linear parabolic systems with delayed measurements

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ABSTRACT
New finite-dimensional adaptive observers are proposed for uncertain heat equation and a class of linear Kuramoto–Sivashinsky equation (KSE) with local output. The observers are based on the modal decomposition approach and use a classical persistent excitation condition to ensure practical exponential convergence of both states and parameters estimation. An important challenge of this work is that it treats the case when the function $\phi_1(\cdot, t)$ of the unknown part in the PDE model depends on the spatial variable and $\phi_1(\cdot, t) \in L^2(0, 1)$.

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Linear parabolic systems; adaptive observer design; modal decomposition

1. Introduction

Adaptive state observers are used to deal with online states and parameters estimation. For finite-dimensional systems, results in this topic can be traced back to the beginning of the 1970s. The most important works in this area can be founded in Kreisselmeier (1977), Zhang (2002), Besançon (2000) and Carroll and Lindorff (1973) and references therein. For distributed parameter systems (DPSs), two approaches exist in the literature. The first one is based on the infinite dimensional model and uses the semigroup theory according to the initial work of Demetriou et al. (1987) and Baumeister and Scondo (1997). Recently, further results based on backstepping technics and Lyapunov analysis (see Ahmed-Ali et al., 2016; Ghousein et al., 2020; Lailler et al., 2024; Smyshlyaev & Krstic, 2006 and references therein) have been also proposed for both parabolic and hyperbolic systems. It has also to be noted that a recent work combining backstepping and a novel methodology based on polynomial optimisation and sum-of-squares decomposition have been also proposed for parabolic systems in Ascencio (2017). The common feature of the above observers is that they are governed by partially differential equations (PDEs). This fact implies that their implementation employs space discretisation methods which may become computationally very hard.

The second approach is based on an abstract model of DPSs. The main advantage of this approach is that it uses only a reduced model of PDE which is described by a set of ODEs. In Lilly (1993), the authors extended the result of adaptive finite dimensional observer (Kreisselmeier, 1977) to a class of infinite-dimensional systems and showed that a reduced-order model (a finite-dimensional part) can be used to identify parameters and states of an infinite-dimensional system. More specially, it is shown that if the residual energy from the unmodelled dynamics is bounded over a finite interval and the input is persistently exciting, then the estimation will be bounded.

In this paper, we propose new finite-dimensional adaptive observers for linear 1D heat equation with additive uncertain parameters based on a reduced model for heat equation. The main difference between Lilly (1993) and this paper is in the fact that we do not consider here that the residual part is exponentially stable and the unknown parameter is involving in the residual part. In fact the presence of uncertainties depending on the spatial variable in the state equation involves the existence of unknown coupling terms which disturb the exponential convergence of the residual part (infinite-dimensional part). We also treat the systems with a local output which is not the case in Lilly (1993). We use a decoupling transformation which removes the uncertainties from the estimation error equation of the finite dimensional part. The main difference between this submission and the paper (Lailler et al., 2024) is in the fact that the proposed observer in Lailler et al. (2024) is described by a PDE which is more complicated in the implementation than the one proposed in this submission which are only a set of finite number of ODEs. Of course, here we have a practical convergence on the estimation error systems but the accuracy can be tuned by a number (N) which represents the number of ODEs involved in the observer's model. Furthermore, for particular cases, our observer, which is constituted by two finite dimensional parts, ensures exponential convergence of the estimations errors with a fixed number of observer's gains which corresponds to the number of unstable modes. Furthermore we show that for sufficiently small varying delay, the result obtained for delay-free case can be extended to the case where the output is subject to a fast delay. Some preliminaries results of this paper will be presented on Ahmed-Ali et al. (2023). Differently from Ahmed-Ali et al. (2023), we treat here a more complicated

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situation by considering uncertainties in both state and output equations, contrarily to Ahmed-Ali et al. (2023), where the uncertainties are considered only in the output equation. We also extend this approach to linear Kuramoto–Sivashinsky equation (KSE) in Section 4.

Notations: We denote with $C(a,b)$ the linear space of continuous functions on the domain (a,b). $L^2(a,b)$ is the linear space of square integrable functions on (a,b). $H^1(a,b)$ is the Sobolev space of functions in $L^2(a,b)$ such that the function and its first derivative have finite L^2 norm. Finally we denote $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

2. System description and assumptions

Consider the class of parabolic system:

$$u_t = u_{xx} + qu + \phi_1(x,t)\theta_1 \quad \text{for } t>0,\ x\in(0,1) \tag{1a}$$

$$u_x(0) = u(1) = 0 \tag{1b}$$

$$u(x,0) = u_0 \tag{1c}$$

with initial condition $u_0 \in H^1(0,1)$ with $u_0(1) = 0$ and under the measurement

$$y(t) = u(0,t) + \phi(t)\theta_2. \tag{2}$$

The constant q is a positive known parameter, ϕ and ϕ_1 are known and continuous functions which satisfy

$$|\phi(t)| \le M_\phi \quad \forall\, t \ge 0 \tag{3}$$

and

$$|\phi_1(x,t)| \le M_{\phi_1} \quad \forall\, x\in[0,1],\ t\ge 0 \tag{4}$$

with some positive constants M_ϕ and M_{ϕ_1}.

We also suppose that $\phi_1(.,t) \in H^1(0,1)$ with $\phi_1(1,t) = 0$. The vectors $\theta_1 \in \mathbf{R}^m$ and $\theta_2 \in \mathbf{R}^p$ are vectors of unknown parameters. The rows $\phi_1(\cdot,t)$ and $\phi(\cdot)$ satisfy respectively $\phi_1^T(\cdot,t) \in \mathbf{R}^m$, $\phi^T(\cdot) \in \mathbf{R}^p$. The term $\phi(t)\theta_2$ models either sensors uncertainties or faults to be detected and isolated. This uncertain term induces a difference between $u(0,t)$ and the available measurement $y(t)$. The term $\phi_1(x,t)\theta_1$ models uncertainties or disturbances which also disturb the model. The role of the adaptive observer is to provide an accurate estimation of both unmeasurable state $u(x,t)$ and the unknown vector of parameters θ.

The well-known regular Sturm–Liouville eigenvalue problem $\psi''(x) + \lambda\psi(x) = 0, x\in[0,1]$ with $\psi(1) = \psi'(0) = 0$, generates an increasing sequence of eigenvalues $\lambda_n = \frac{\pi^2}{4}(2n-1)^2$ $n \ge 1$ with corresponding eigenfunctions $\psi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x)$ for $n \ge 1$. The eigenfunctions ψ_n form an orthonormal basis of $L^2(0,1)$. A strong solution of (1a) is a function $u \in L^2((0,\infty); H^2(0,1)) \cap C([0,\infty); H^1(0,1))$ and $u_t \in L^2((0,\infty); L^2(0,1))$ that satisfies (1c) for $t = 0$ and (1a), (1b) for almost all $t > 0$. By (Robinson, 2001, Theorem 7.7), (1a) has a unique strong solution for $u_0 \in H^1(0,1)$ s.t $u_0(1) = 0$. Consequently the solutions of Equation (1a) can be presented as

$$u(x,t) = \sum_{n=1}^{\infty} z_n(t)\psi_n(x) \tag{5}$$

where $z_n(t) = \int_0^1 u(x,t)\psi_n(x)\,dx$. The function ϕ_1 can be also written as follows:

$$\phi_1(x,t) = \sum_{n=1}^{\infty} p_n(t)\psi_n(x) \tag{6}$$

with $p_n(t) = \int_0^1 \phi_1(x,t)\psi_n(x)\,dx$. Note that since both ϕ_1 and ψ_n are bounded, then p_n are also bounded and satisfy $|p_n| \le \sqrt{2}M_{\phi_1}$.

Differentiating the modes z_n and further integrating by parts twice we have

$$\dot{z}_n(t) = -\lambda_n z_n(t) + q z_n(t) + p_n(t)\theta_1 \quad n = 1,2,\ldots. \tag{7}$$

The output y can also be expressed as follows:

$$y(t) = \sqrt{2}\sum_{n=1}^{\infty} z_n(t) + \phi(t)\theta_2 \tag{8}$$

Since λ_n is an increasing sequence then we can define an integer N_0 as the smallest integer n for which the following inequality holds:

$$-\lambda_n + q < 0, \quad \forall\, n > N_0 \tag{9}$$

We assume additionally that $q \ne \lambda_n$.

Remark 2.1: Note that the presence of the uncertain term $\phi_1(x,t)\theta_1$ in the state equation induces that the infinite dimensional part constituted by the modes $z_i, i = N_0 + 1\cdots$. is not exponentially stable, but it is ultimately bounded.

3. Finite-dimensional adaptive observer design

3.1 Adaptive observer structure

Following Katz Fridman (2020), we will construct N-dimensional adaptive observer with $N \ge N_0$ to be defined later. We propose the following structure :

$$\begin{cases} \hat{u}(x,t) = \sum_{n=1}^{N} \hat{z}_n(t)\psi_n(x) \\ \dot{\hat{z}}_n(t) = -\mu_n\hat{z}_n + p_n(t)\hat{\theta}_1 - l_n(\hat{y} - y) + v_1 \quad n = 1,\ldots,N_0 \\ \dot{\hat{z}}_n(t) = -\mu_n\hat{z}_n + p_n(t)\hat{\theta}_1 + v_2 \quad n = N_0+1,\ldots,N \\ \hat{y} = \sqrt{2}\sum_{n=1}^{N}\hat{z}_n(t) + \phi(t)\hat{\theta}_2. \end{cases} \tag{10}$$

with N_0 and N are respectively defined in (9) and (22), $\mu_n = \lambda_n - q$, l_n are observer gains, $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$ are the estimates of θ_1 and θ_2, v_1 and v_2 are additional signals that we will choose later on.

Denote $A_n = diag(-\mu_1,\ldots,-\mu_n)$, $C_n = (\sqrt{2},\ldots,\sqrt{2})$ is a row which has n columns, $L_n = (l_1,\ldots,l_n)^T$ and, $p_n = (p_1,\ldots,p_n)^T$ are vectors with n raws. $A_{i-j} = diag(-\mu_{i+1},\ldots,-\mu_j)$, $p_{i-j} = (p_i,\ldots,p_j)^T$, $C_{n-j} = (\sqrt{2},\ldots,\sqrt{2})$ is a row which has $n-j$ columns. Since (A_{N_0}, C_{N_0}) is observable (Katz & Fridman, 2020), we choose L_{N_0} such that $A_{N_0} - L_{N_0}C_{N_0}$ is Hurwitz.

Consider the state estimation errors $\tilde{Z}_{N_0} = (\tilde{z}_1, \ldots, \tilde{z}_{N_0})^T$ and $\tilde{Z}_{N-N_0} = (\tilde{z}_{N_0+1}, \ldots, \tilde{z}_N)^T$ where $\tilde{z}_i = \hat{z}_i - z_i$, and the estimation parameters errors $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$ and $\tilde{\theta}_2 = \hat{\theta}_2 - \theta_2$. Then the observation error system is expressed as follows:

$$\begin{aligned}\dot{\tilde{Z}}_{N_0}(t) &= (A_{N_0} - L_{N_0}C_{N_0})\tilde{Z}_{N_0} \\ &\quad + p_{N_0}\tilde{\theta}_1(t) - L_{N_0}\phi(t)\tilde{\theta}_2(t) \\ &\quad - Ł_{N_0}C_{N-N_0}\tilde{Z}_{N-N_0} + L_{N_0}\sqrt{2}\zeta(t) + v_1 \\ \dot{\tilde{Z}}_{N-N_0}(t) &= A_{N-N_0}\tilde{Z}_{N-N_0} + p_{N-N_0}(t)\tilde{\theta}_1(t) + v_2 \\ \hat{u}(x,t) - u(x,t) &= \sum_{i=1}^{N} \tilde{z}_i(t)\psi_i(x) - \sum_{i\geq N+1} z_i(t)\psi_i(x) \end{aligned} \tag{11}$$

where

$$\zeta(t) = \sum_{i=N+1}^{\infty} z_i(t) \tag{12}$$

It was already proven in Katz and Fridman (2020) that $\sum_{n=N+1}^{\infty} \lambda_n z_n^2$ is well defined for any $t > t_0$ and $\sum_{n=N+1}^{\infty} \lambda_n z_n^2 \leq ||u_x(.,t)||^2_{L^2(0,1)}$. We have also the following property :

$$|\zeta(t)|^2 = \left|\sum_{i=N+1}^{\infty} z_i(t)\right|^2 \leq \sum_{n=N+1}^{\infty} \lambda_n z_n^2 \tag{13}$$

Consider the Lyapunov function (Katz & Fridman, 2020),

$$V(t) = \sum_{n=N+1}^{\infty} \lambda_n z_n^2 \tag{14}$$

then its time derivative will be written as follows:

$$\dot{V}(t) = -2\sum_{n=N+1}^{\infty} \lambda_n\mu_n z_n^2 + 2\sum_{n=N+1}^{\infty} \lambda_n z_n p_n(t)\theta_1 \tag{15}$$

Using the Young's inequality, we derive

$$\begin{aligned}\dot{V}(t) &\leq -2\sum_{n=N+1}^{\infty} \lambda_n\mu_n z_n^2 + 2\sum_{n=N+1}^{\infty} \lambda_n z_n^2 \\ &\quad + \frac{|\theta_1|^2}{2}\sum_{n=N+1}^{\infty} \lambda_n p_n^2(t). \end{aligned} \tag{16}$$

On the other hand, for any positive constant δ we can write

$$\begin{aligned}\dot{V} + 2\delta V &\leq -2\sum_{n=N+1}^{\infty} \lambda_n\mu_n z_n^2 + 2\sum_{n=N+1}^{\infty} \lambda_n z_n^2 \\ &\quad + 2\delta\sum_{n=N+1}^{\infty} \lambda_n z_n^2 + \frac{|\theta_1|^2}{2}\sum_{n=N+1}^{\infty} \lambda_n p_n^2 \end{aligned} \tag{17}$$

Since $\phi_1(\cdots,t) \in H^1(0,1)$ with $\phi_1(1,t) = 0$, then $\sum_{n=N+1}^{\infty} \lambda_n p_n^2 \leq ||\phi_1'||^2_{L^2(0,1)}$. From this we deduce the following inequality:

$$\begin{aligned}\dot{V} + 2\delta V &\leq -2\sum_{n=N+1}^{\infty} (\mu_n\lambda_n - \delta\lambda_n - \lambda_n)z_n^2 \\ &\quad + \frac{|\theta_1|^2}{2}||\phi_1'||^2_{L^2(0,1)} \end{aligned} \tag{18}$$

Choosing N sufficiently large such that

$$\mu_n - \delta - 1 > 0, \quad \forall\, n > N \tag{19}$$

which gives us

$$N > \frac{1}{\pi}\sqrt{q+1+\delta} + \frac{1}{2} \tag{20}$$

then

$$\dot{V} \leq -2\delta V + \frac{|\theta_1|^2}{2}||\phi_1'||^2_{L^2(0,1)} \tag{21}$$

Since $\phi_1(.,t) \in H^1(0,1)$ with $\phi_1(1,t) = 0$ and $u_0 \in H^1(0,1)$ with $u_0(1) = 0$, then both $V(0)$ and $||\phi_1'||_{L^2(0,1)}$ are bounded and consequently, we deduce that

$$|\zeta(t)|^2 \leq e^{-2\delta t}V(0) + \frac{|\theta_1|^2}{4\delta}||\phi_1'||^2_{L^2(0,1)}. \tag{22}$$

From this last inequality, we deduce that $|\zeta(t)|$ converges to a ball with a radius which is proportional to $\frac{|\theta_1|}{2\sqrt{\delta}}||\phi_1'||_{L^2(0,1)}$. This means that by increasing δ, the above radius can be made as small as we want. Note that if δ is increased, then N must be also increased following condition (22).

Remark 3.1: The Number N is required to be sufficiently large for two reasons: the first one is related to the number of unstable modes (N_0 defined in (9)). The observer should at least have this minimum number of ODEs to ensure the boundedness of the estimation error. The second reason is related to the accuracy of the observer. As we can see the accuracy of the estimation errors depends on the positive constant δ. This is the main sense of Equation (19) which describes a sufficient condition involving both N (number of ODEs) and δ. More precisely, we use this inequality to derive a Number N that ensures a desired accuracy of our observer. It has also to be noted that using PDE is equivalent to have an infinite number of ODEs.

Consider for $\tilde{Z}_{N_0}$ and $\tilde{Z}_{N-N_0}$ defined in (11) the decoupling transformations (Zhang, 2002)

$$e_{N_0}(t) = \tilde{Z}_{N_0}(t) - \alpha_1^1(t)\tilde{\theta}_1(t) - \alpha_1^2(t)\tilde{\theta}_2(t) \tag{23}$$

and

$$e_{N-N_0}(t) = \tilde{Z}_{N-N_0}(t) - \alpha_2(t)\tilde{\theta}_1(t) \tag{24}$$

where α_1^1 and α_1^2 are the solutions of an auxiliary filter which is defined as follows:

$$\begin{cases}\dot{\alpha}_1^1(t) = (A_{N_0} - L_{N_0}C_{N_0})\alpha_1^1(t) + p_{N_0} - L_{N_0}C_{N-N_0}\alpha_2(t) \\ \dot{\alpha}_1^2(t) = (A_{N_0} - L_{N_0}C_{N_0})\alpha_1^2(t) - L_{N_0}\phi(t) \\ v_1 = \alpha_1^1(t)\dot{\hat{\theta}}_1 + \alpha_1^2(t)\dot{\hat{\theta}}_2 \end{cases} \tag{25}$$

and α_2 is the solution of an auxiliary filter which is also defined as follows:

$$\begin{cases}\dot{\alpha}_2(t) = A_{N-N_0}\alpha_2(t) + p_{N-N_0}(t) \\ v_2 = \alpha_2(t)\dot{\hat{\theta}}_1 \end{cases} \tag{26}$$

From this, we deduce two ODEs of e_{N_0} and e_{N-N_0} which do not depend on $\tilde{\theta}$.

$$\dot{e}_{N_0}(t) = (A_{N_0} - L_{N_0}C_{N_0})e_{N_0}(t) + \sqrt{2}\,L_{N_0}\zeta(t) - L_{N_0}C_{N-N_0}e_{N-N_0}(t). \quad (27)$$

and

$$\dot{e}_{N-N_0}(t) = (A_{N-N_0})e_{N-N_0}(t). \quad (28)$$

Remark 3.2: When we write the state estimation errors equations (11), we remark that they also depend on the parameter estimation errors. The auxiliary filters (25) and (26) are then used to decouple the state estimation errors from the parameter estimation errors. As you can see the parameter estimation errors is not involved in Equations (27) and (28). We will also see below that these filters will facilitate greatly the convergence analysis and the design of parameter estimation laws.

Since $N > N_0$, then A_{N-N_0} is Hurwitz and consequently e_{N-N_0} is exponentially stable. Consider

$$V_0(t) = e_{N_0}(t)^T P_0 e_{N_0}(t) \quad (29)$$

where P_0 satisfies the following inequality:

$$P_0(A_{N_0} - L_{N_0}C_{N_0}) + (A_{N_0} - L_{N_0}C_{N_0})^T P_0 \leq -2P_0 \quad (30)$$

Its time derivative will be expressed as follows:

$$\dot{V}_0 \leq -2V_0 + 2e_{N_0}(t)^T P_0\left(\sqrt{2}\,L_{N_0}\zeta(t) - L_{N_0}C_{N-N_0}e_{N-N_0}(t)\right) \quad (31)$$

Applying again Young's inequality, then

$$\dot{V}_0 \leq -2V_0 + \epsilon_1|e_{N_0}(t)|^2 + \frac{4}{\epsilon_1}|P_0|^2||L_{N_0}|^2|\zeta(t)|^2 + \frac{2|P_0|^2}{\epsilon_1}|L_{N_0}C_{N-N_0}|^2|e_{N-N_0}(t)|^2 \quad (32)$$

which gives us

$$\dot{V}_0 \leq -\left(2 - \frac{\epsilon_1}{\lambda_{min}(P_0)}\right)V_0 + \frac{4}{\epsilon_1}|P_0|^2||L_{N_0}|^2|\zeta(t)|^2 \quad (33)$$

$$+ \frac{2|P_0|^2}{\epsilon_1}|L_{N_0}C_{N-N_0}|^2|e_{N-N_0}(t)|^2. \quad (34)$$

Choosing $\epsilon_1 = \lambda_{min}(P_0)$, then

$$\dot{V}_0 \leq -V_0 + \frac{4}{\lambda_{min}(P_0)}|P_0|^2||L_{N_0}|^2|\zeta(t)|^2 \quad (35)$$

$$+ \frac{2|P_0|^2}{\lambda_{min}(P_0)}|L_{N_0}C_{N-N_0}|^2|e_{N-N_0}(t)|^2. \quad (36)$$

Since $e_{N-N_0}(t)$ is exponentially vanishing and $|\zeta(t)|$ is bounded, we can also deduce from the comparison lemma that $|e_{N_0}|$ converges to a ball with a radius which is proportional to $\frac{|\theta_1|}{2\sqrt{\delta}}||\phi_1'||_{L^2(0,1)}$.

3.1.1 Estimation law design

Following Zhang (2002), we propose the following estimation law:

$$\begin{pmatrix}\dot{\hat{\theta}}_1\\ \dot{\hat{\theta}}_2\end{pmatrix} = -R(t)K^T(t)(\hat{y}(t) - y(t)) \quad (37)$$

where

$$K(t) = (C_{N_0}\alpha_1^1(t) + C_{N-N_0}\alpha_2(t)\;\; C_{N_0}\alpha_1^2(t) + \phi(t)) \quad (38)$$

and

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = R(t) - R(t)K^T(t)K(t)R(t) \quad (39)$$

It was already proven in Zhang (2002) that if $|\alpha_1^1|$, $|\alpha_1^2|$ and $|\phi|$ are bounded and if the persistent excitation condition

$$\int_t^{t+T} K^T(s)K(s)\,\mathrm{d}s \geq \beta_0\mathbb{I} \quad (40)$$

holds for some positive constant β_0, then both $R(t)$ and $R^{-1}(t)$ are positive definite matrices and there exist two positive constants β_1 and β_2 such that the following inequalities hold :

$$\beta_1\mathbb{I}_m \leq R(t) \leq \beta_2\mathbb{I}_m \quad (41)$$

and the inverse matrix satisfies

$$\frac{\mathrm{d}R^{-1}(t)}{\mathrm{d}t} = -R^{-1}(t) + K^T(t)K(t) \quad (42)$$

with

$$\beta_1\mathbb{I}_m \leq R^{-1}(t) \leq \beta_2\mathbb{I}_m \quad (43)$$

3.1.2 Convergence analysis

The parameter estimation error is governed by the following ODEs :

$$\dot{\tilde{\theta}} = -R(t)K^T(t)K(t)\tilde{\theta}(t) - R(t)K^T(t)(C_{N_0}\epsilon_{N_0}(t) + C_{N-N_0}e_{N-N_0}(t) - \sqrt{2}\zeta(t)). \quad (44)$$

To study the convergence of the vector $\tilde{\theta} = \begin{pmatrix}\tilde{\theta}_1\\ \tilde{\theta}_2\end{pmatrix}$, let us consider the following Lyapunov function for (44):

$$V_\theta = \tilde{\theta}^T R^{-1}(t)\tilde{\theta}. \quad (45)$$

Then the time-derivative of V_θ satisfies the following equality:

$$\dot{V}_\theta(t) = -V_\theta(t) + |C_{N_0}\epsilon_{N_0}(t) + C_{N-N_0}e_{N-N_0}(t) - \sqrt{2}\zeta(t)|^2. \quad (46)$$

Using Young's inequality, we derive

$$\dot{V}_\theta(t) \leq -V_\theta(t) + 4|C_{N_0}\epsilon_{N_0}(t)|^2 + 4|C_{N-N_0}e_{N-N_0}(t)|^2 + 8|\zeta(t)|^2. \quad (47)$$

Since e_{N-N_0} converges exponentially to zero and both e_{N_0} and ζ converge exponentially to balls, then by applying the comparison lemma to (47), we conclude that $\tilde{\theta}$ will also converge to a ball with a radius which is proportional to $\frac{|\theta|}{\sqrt{\delta}}||\phi_1'||_{L^2(0,1)}$. On the other hand, from (23) and (24), we can deduce that $|\tilde{Z}_N(t)|^2 \le 4|\epsilon_N(t)|^2 + 4|\alpha(t)|^2|\tilde{\theta}(t)|^2$, where $\alpha = (\alpha_1^1, \alpha_1^2, \alpha_2)^T$. Since both systems (25) and (26) are ISS and $|p_N|$ and $|\phi|$ are bounded, then $|\alpha_1^1|$, $|\alpha_1^2|$ and $|\alpha_2|$ are also bounded. This allows us to conclude that $|\tilde{Z}_N|$ is also exponentially convergent to a ball with a similar radius. Now if we use the Parseval's equality, then we have

$$||\hat{u}(\cdot,t) - u(\cdot,t)||^2_{H^1(0,1)} = \sum_{i=1}^{N} \lambda_i \tilde{z}_i^2(t) - \sum_{n\ge N+1} \lambda_n z_n^2(t). \quad (48)$$

Since $\sum_{i=1}^{N} \lambda_i \tilde{z}_i^2(t) \le \lambda_N |\tilde{Z}_N|^2$ then we can also conclude that the H^1 norm $||\hat{u}(.,t) - u(.,t)||_{H^1(0,1)}$ converges to a ball with a radius which is proportional to $\frac{|\theta|}{\sqrt{\delta}}||\phi_1'||_{L^2(0,1)}$.

Theorem 3.1: *Consider system* (1a) *with initial condition $u_0 \in H^1(0,1), u_0(1) = 0$, and adaptive observer described by* (10), (25), (26) *and* (37). *Given $\delta > 0$, let $N_0 \in \mathbb{N}$ satisfy* (9) *and $N \in \mathbb{N}$ satisfy* (22). *Let the vector of gains $L_{N_0} = (l_1,\ldots,l_{N_0})^T$ satisfy* (30). *Then under persistent excitation condition* (40), *the norms $||\tilde{u}(.,t)||_{H^1(0,1)}$ and $|\tilde{\theta}_1(t)|$ and $|\tilde{\theta}_2(t)|$ converge to balls with a radius which is proportional to $\frac{|\theta_1|}{\sqrt{\delta}}||\phi_1'||_{L^2(0,1)}$.*

Remark 3.3: It has to be noted that the radius $\frac{|\theta_1|}{\sqrt{\delta}}||\phi_1'||_{L^2(0,1)}$ can be made as small as we want by increasing δ. This also implies that N has to be sufficiently large following (22).

Remark 3.4: Note that if the function ϕ_1 can be expressed as a finite sum of the form,

$$\phi_1(x,t) = \sum_{n=1}^{N_{\phi_1}} p_n(t)\psi_n(x) \quad (49)$$

then by tacking $N \ge N_{\phi_1}$, we deduce that $p_i = 0, \forall i > N$ and consequently both $||\tilde{u}(.,t)||_{L^2(0,1)}$ and $|\tilde{\theta}(t)|$ converge exponentially to zero.

3.2 Extension to the case of delayed output

In this section, we extend the above results to the case of delayed output with known and bounded fast varying delay $\tau(t) \ge 0$ (without any constraints on the delay derivative). In this case

$$y(t) = u(0, t-\tau(t)) + \phi(t-\tau(t)))\theta_2. \quad (50)$$

Following (10), we propose the following structure:

$$\begin{cases} \hat{u}(x,t) = \sum_{n=1}^{N} \hat{z}_n(t)\psi_n(x) \\ \dot{\hat{z}}_n(t) = -\mu_n\hat{z}_n + p_n(t)\hat{\theta}_1 - l_n(\hat{y} - y) + v_1 \quad n = 1,\ldots,N_0 \\ \dot{\hat{z}}_n(t) = -\mu_n\hat{z}_n + p_n(t)\hat{\theta}_1 + v_2 \quad n = N_0+1,\ldots,N \\ \hat{y} = \sqrt{2}\sum_{n=1}^{N} \hat{z}_n(t-\tau(t)) + \phi(t-\tau(t))\hat{\theta}_2. \end{cases} \quad (51)$$

Using the same decoupling transformations (23) and (24) where α_1^1 and α_1^2 the solution of an auxiliary filter which is defined as follows:

$$\begin{cases} \dot{\alpha}_1^1(t) = A_{N_0}\alpha_1^1(t) - L_{N_0}C_{N_0}\alpha_1^1(t-\tau(t)) + p_{N_0} \\ \qquad - L_{N_0}C_{N-N_0}\alpha_2(t-\tau(t)) \\ \dot{\alpha}_1^2(t) = A_{N_0}\alpha_1^2(t) - L_{N_0}C_{N_0}\alpha_1^2(t-\tau(t)) - L_{N_0}\phi(t-\tau(t)) \\ v_1 = \alpha_1^1(t)\dot{\hat{\theta}}_1 + \alpha_1^2(t)\dot{\hat{\theta}}_2 + L_{N_0}C_{N_0}\alpha_1^1(t-\tau(t)) \\ \qquad \times (\hat{\theta}_1(t-\tau(t)) - \hat{\theta}_1(t)) + L_{N_0}C_{N-N_0}\alpha_2(t-\tau(t)) \\ \qquad \times (\hat{\theta}_1(t-\tau(t)) - \hat{\theta}_1(t)) + L_{N_0}\phi(t-\tau(t)) \\ \qquad \times (\hat{\theta}_2(t-\tau(t)) - \hat{\theta}_2(t)) \end{cases} \quad (52)$$

and α_2 is the solution of an auxiliary filter which is also defined as follows:

$$\begin{cases} \dot{\alpha}_2(t) = A_{N-N_0}\alpha_2(t) + p_{N-N_0}(t) \\ v_2 = \alpha_2(t)\dot{\hat{\theta}}_1 \end{cases} \quad (53)$$

we obtain the following DDEs for e_{N_0} and e_{N-N_0} which do not depend on $\tilde{\theta}$.

$$\dot{e}_{N_0}(t) = A_{N_0}e_{N_0}(t) - L_{N_0}C_{N_0}e_{N_0}(t-\tau(t)) \quad (54)$$
$$+ \sqrt{2}L_{N_0}\zeta(t-\tau(t))$$
$$- L_{N_0}C_{N-N_0}e_{N-N_0}(t-\tau(t)) \quad (55)$$

and

$$\dot{e}_{N-N_0}(t) = (A_{N-N_0})e_{N-N_0}(t) \quad (56)$$

where $\zeta(t)$ is defined in (12).

Inspired by Ahmed-Ali et al. (2020), we propose the following adaptive law:

$$\begin{cases} \dot{\hat{\theta}} = R_\tau(t)K^T(t-\tau(t))(\hat{y}(t) - y(t)) + R_\tau(t)K^T(t-\tau(t)) \\ \qquad \times h(t-\tau(t))\hat{\theta}(t-\tau(t)) - \hat{\theta}(t)) \\ h(t-\tau(t)) = [C_{N_0}\alpha_1^1(t-\tau(t)) + C_{N-N_0}\alpha_2(t-\tau(t)) \\ \quad C_{N_0}\alpha_1^2(t-\tau(t)) + \phi(t)] \\ \frac{\mathrm{d}R_\tau(t)}{\mathrm{d}t} = R_\tau(t) - R_\tau(t)K^T(t-\tau(t))K(t-\tau(t))R_\tau(t) \end{cases} \quad (57)$$

where $\hat{\theta} = \begin{pmatrix}\hat{\theta}_1 \\ \hat{\theta}_2\end{pmatrix}$ and $K(t)$ given by (38). The parameter estimation error $\tilde{\theta} = \hat{\theta} - \theta$ is governed by the following ODE:

$$\dot{\tilde{\theta}} = -R_\tau(t)K^T(t-\tau(t))K(t-\tau(t))\tilde{\theta}(t) - R_\tau(t)K^T(t-\tau(t))$$
$$\times (C_N\epsilon_N(t-\tau(t)) - \sqrt{2}\zeta(t-\tau(t))) \quad (58)$$

Note that Equation (54) is very similar to (22) of Selivanov and Fridman (2018) but it contains two supplementary terms. An exponential vanishing one which is $e_{N-N_0}(t)$ and the bounded term $\zeta(t-\tau(t))$. Let us consider the following Lyapunov functional (Selivanov & Fridman, 2018):

$$W_d = e_{N_0}^T P_0 e_{N_0} + \rho_1 \int_{t-\tau_m}^{t} |e_{N_0}(s)|^2 \, ds + \tau_m \rho_2 \int_{t-\tau_m}^{0} \int_{t+\theta}^{t} e^{-2\delta(t-s)} |\dot{e}_{N_0}(s)|^2 \, ds$$

where ρ_1 and ρ_2 are two positive constants. Then, from Theorem 1 of Selivanov and Fridman (2018), we ensure the existence of a bound τ_m of the delay and a positive constant γ such that

$$\dot{W}_d \leq -2\delta W_d + \gamma\left(|\xi(t-\tau(t))|^2 + |e_{N-N_0}|^2\right).$$

From this we deduce that both (54) and (58) are exponentially convergent to balls with a radius similar to one of Theorem 3.1.

3.3 Extension to sampled-data case

We suppose that the output

$$y(t_k) = u(0,t_k) + \phi(t_k)\theta_2 \tag{59}$$

is available only at sampling instants t_k which constitute an increasing sequence defined as follows: $0 = t_0 < t_1 < \cdots, < t_k < \cdots, \lim_{t_k\to\infty} = \infty$ with $t_{k+1} - t_k \leq h$ is the maximum allowable sampling period. It is well known that the sampled-data case can be reformulated as time-delay one with a delay $\tau(t) = t - t_k$ for all $t \in [t_k, t_{k+1})$. Then the observer for the sampled-data case can be obtained by replacing $t - \tau(t)$ by t_k in Equations (51), (52), (53), (57).

4. Adaptive observer for linear KSE

In this section, we show that we can also propose a finite-dimensional adaptive observer for linear Kuramoto–Sivashinsky equation (KSE). As in Katz and Fridman (2021) consider the KSE

$$w_t = -w_{xxxx} - \nu w_{xx} + \phi_1^s(x,t)\theta_1, \quad \text{for } t>0,\ x\in(0,1) \tag{60a}$$

$$w(0,t) = 0, \quad w(1,t) = 0, \quad w_{xx}(0,t) = w_{xx}(1,t) = 0 \tag{60b}$$

$$w(x,0) = w_0 \tag{60c}$$

with $w_0 \in H^2(0,1)$ and under the measurement

$$y(t) = w(x^*,t) + \phi^s(t)\theta_2 \quad x^* \in (0,1). \tag{61}$$

where the functions ϕ_1^s and ϕ^s are known functions and bounded. We also suppose that $\phi_1^s(.,t) \in H_0^1(0,1)$. A strong solution of (60a) is a function $w \in L^2((0,\infty); H^4(0,1)) \cap C([0,\infty); H^2(0,1))$ and $w_t \in L^2((0,\infty); L^2(0,1))$ that satisfies (60c) for $t=0$ and (60a), (60b) for almost all $t>0$. By [17, Th. 7.7], (60a) has a unique strong solution for $w_0 \in H^2(0,1)$ s.t $w_0(1) = w_0(0) = 0$. We suppose that $\nu \neq \pi^2 n^2, \forall n \geq 1$. By using eigenvalues and eigenfunctions of the Sturm–Liouville problem $\varphi'' + \lambda\varphi = 0, \varphi(0) = \varphi(1) = 0$ given by $a_n = \pi^2 n^2$ and $\varphi_n(x) = \sqrt{2}\, sin(\sqrt{a_n}x)$, the solution of (60a) can be presented as

$$w(x,t) = \sum_{n\geq 1} w_n(t)\varphi_n(x) \tag{62}$$

where w_n are solutions of the following ODEs:

$$\dot{w}_n(t) = -(a_n^2 - \nu a_n)w_n(t) + p_n^s(t)\theta_1 \quad n = 1,2,\ldots. \tag{63}$$

with $w_n(t) = \int_0^1 w(x,t)\varphi_n(x)\,dx$ and $p_n^s(t) = \int_0^1 \phi_1^s(x,t)\varphi_n(x)\,dx$. The measurement y will be also expressed as follows:

$$y(t) = \sqrt{2}\sum_{n\geq 1} w_n(t)\sin(\sqrt{a_n}x^*) + \phi^s(t)\theta_2. \tag{64}$$

Note that since $x^* \in (0,1)$, then $\sin(\sqrt{a_n}x^*) \neq 0, \forall n \geq 1$. As for reaction–diffusion systems, we can propose under the persistent excitation condition (40), the following N^s finite-dimensional adaptive observer structure where $N^s \geq N_0^s$ to be defined later:

$$\begin{cases} \hat{w}(x,t) = \sum_{n=1}^{N^s} \hat{w}_n(t)\varphi_n(x) \\ \dot{\hat{w}}_n(t) = -(a_n^2 - \nu a_n)\hat{w}_n + +p_n^s(t)\hat{\theta}_1(t) - l_n^s(\hat{y} - y(t)) + v_1^s \\ \quad n = 1,\ldots,N_0^s \\ \dot{\hat{w}}_n(t) = -(a_n^2 - \nu a_n)\hat{w}_n + +p_n^s(t)\hat{\theta}_1(t) + v_2^s \\ \quad n = N_0^s + 1,\ldots,N^s \\ \hat{y}(t) = \sqrt{2}\sum_{n=1}^{N^s} \sin(\sqrt{a_n}x^*)\hat{w}_n(t) + \phi(t)\hat{\theta}_2(t) \\ \begin{pmatrix} \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \end{pmatrix} = -R^s(t)(K^s)^T(t)(\hat{y}(t) - y(t)) \end{cases} \tag{65}$$

The integer N_0^s is the smallest positive constant satisfying

$$a_{N_0^s} - \nu > 0 \tag{66}$$

Consider the observation error $\tilde{W}_{N^s} = (\tilde{w}_1,\ldots,\tilde{w}_{N^s})^T$ where $\tilde{w}_i = \hat{w}_i - w_i$, and the parameter estimation error $\tilde{\theta} = \hat{\theta} - \theta$. Then the observation error system is expresses as follows:

$$\tilde{w}(x,t) = \hat{w}(x,t) - w(x,t) = \sum_{n=1}^{N^s}\tilde{w}_i\varphi_n(x) - \sum_{n\geq N^s+1} w_i\varphi_n(x) \tag{67}$$

$$\dot{\tilde{W}}_{N_0^s}(t) = (A_{N_0^s}^s - L_{N_0^s}^s C_{N_0^s}^s)\tilde{W}_{N_0^s} + p_n^s(t)\tilde{\theta}_1(t) - L_{N_0^s}^s\phi(t)\tilde{\theta}_2(t) - +L_{N_0^s}^s\zeta^s(t) + v_1^s \tag{68}$$

and

$$\dot{\tilde{W}}_{N^s-N_0^s}(t) = A_{N^s-N_0^s}^s\tilde{W}_{N^s-N_0^s} + p_n^s(t)\tilde{\theta}_1(t) + v_2^s \tag{69}$$

where

$$A_{N^s}^s = \text{diag}(-(a_1^2 - \nu a_1),\ldots,-(a_{N^s}^2 - \nu a_{N^s})) \tag{70}$$

and

$$C^s_{N^s} = \sqrt{2}(\sin(\sqrt{a_1}x^*)), \ldots, \sin(\sqrt{a_{N^s}}x^*)). \quad (71)$$

with

$$\zeta^s(t) = \sqrt{2}\sum_{n\geq N^s+1} \sin(\sqrt{a_n}x^*)w_n(t). \quad (72)$$

Since all terms $(a_n^2 - \nu a_n)$ and $\sin(\sqrt{a_n}x^*)$ are non equal to zero, then the pair $(A^s_{N^s}, C^s_{N^s})$ is detectable. This allows us to choose $L^s_{N^s} = (l^s_1, \ldots, l^s_{N^s})^T$ such that the matrix $A^s_{N^s} - L^s_{N^s}C^s_{N^s}$ satisfy for symmetric positive matrix P^s_0 the following inequality:

$$P^s_0(A^s_{N_0} - L^s_{N^s_0}C^s_{N^s_0}) + (A^s_{N_0} - L_{N^s_0}C^s_{N^s_0})^T P^s_0 \leq -2P^s_0. \quad (73)$$

Now if we consider the same decoupling transformations (23) and (24), we obtain

$$\dot{\epsilon}^s_{N^s_0}(t) = (A^s_{N^s_0} - L^s_{N^s_0}C^s_{N^s_0})\epsilon^s_{N^s_0}(t) + L^s_{N^s_0}\zeta^s(t) - L_{N^s_0}C_{N^s-N^s_0}e_{N^s-N^s_0}(t) \quad (74)$$

and

$$\dot{e}^s_{N^s-N^s_0}(t) = (A_{N^s-N^s_0})\, e^s_{N^s-N^s_0}(t) \quad (75)$$

and the parameter estimation error will also satisfy

$$\dot{\tilde{\theta}} = -R^s(t)(K^s)^T(t)K^s(t)\tilde{\theta}(t) - R^s(t)(K^s)^T(t)(C_{N^s_0}\epsilon_{N^s_0}(t) + C_{N^s-N^s_0}\, e^s_{N-N_0}(t) - \sqrt{2}\zeta^s(t)). \quad (76)$$

Note that from Katz and Fridman (2021), we have the following property:

$$|\zeta^s(t)|^2 \leq \sum_{n\geq N^s+1} a_n w_n^2(t) < \infty \quad (77)$$

If we consider the Lyapunov function of ((29) in Katz and Fridman (2022)) :

$$V_s(t) = \sum_{n\geq N^s+1} a_n w_n^2(t) \quad (78)$$

then we have

$$\dot{V}_s(t) + 2\delta V_s(t) = -2\sum_{n\geq N^s+1} a_n^2(a_n - \nu)w_n^2 + \sum_{n\geq N^s+1} 2\delta a_n w_n^2 + 2\sum_{n\geq N^s+1} a_n w_n p^s_n(t)\theta_1 \quad (79)$$

Using young's inequality, we have for any positive constant δ:

$$\dot{V}_s(t) + 2\delta V_s(t) \leq -2\sum_{n\geq N^s+1} [a_n^2(a_n - \nu) - \delta a_n - a_n]w_n^2 + \frac{|\theta_1|^2}{2}\sum_{n\geq N^s+1} a_n(p^s_n(t))^2 \quad (80)$$

Now, choosing N^s as the smallest integer which satisfy the following inequality:

$$a_{N^s}(a_{N^s} - \nu) - \delta - 1 > 0 \quad (81)$$

with $a_{N^s} = \pi^2(N^s)^2$. From this we derive similar property than the heat equation case:

$$\dot{V}_s(t) \leq -2\delta V_s(t) + \frac{|\theta_1|^2}{2}||(\phi^s_1)'||^2_{L^2(0,1)} \quad (82)$$

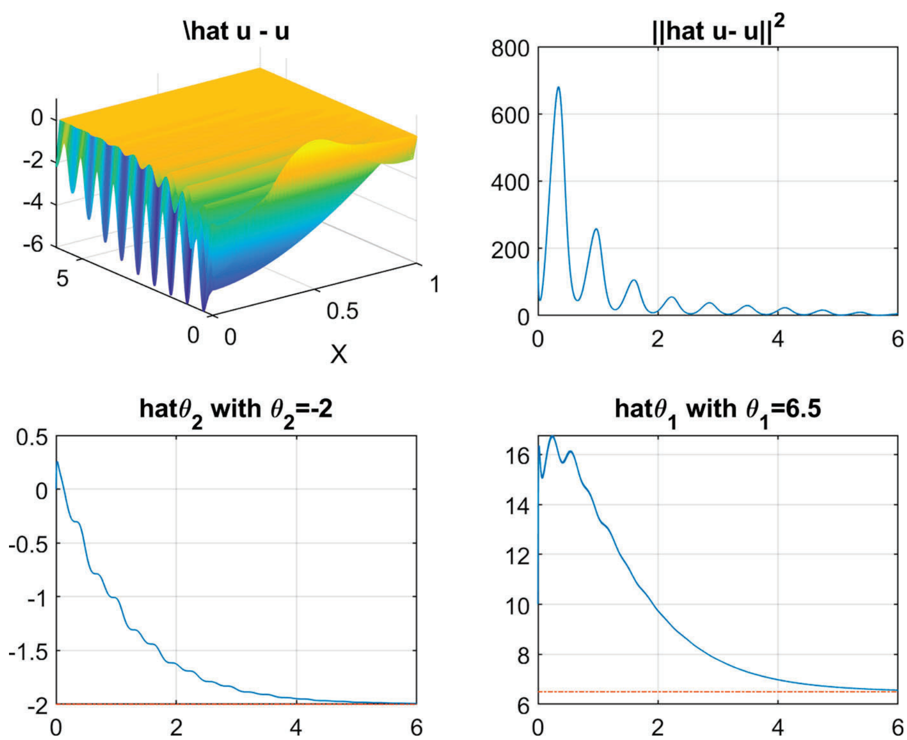

Figure 1. Estimation errors for system (1a) with $\theta_1 = 6.5$ and $\theta_2 = -2$.

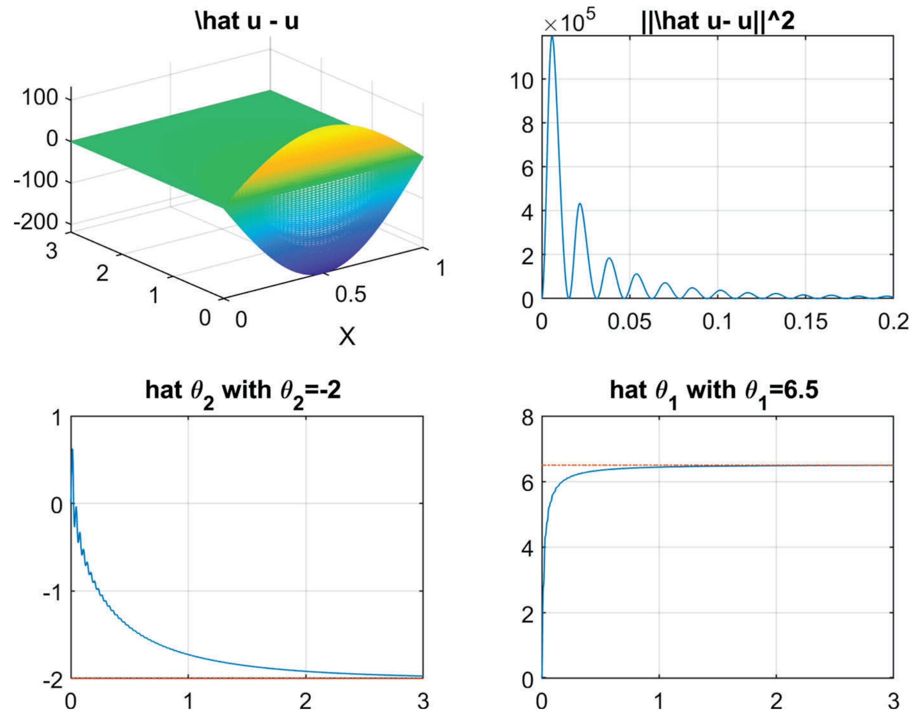

Figure 2. Estimation errors for system (60a) with $\theta_1 = 6.5$ and $\theta_2 = -2$.

which gives us

$$|\zeta^s(t)|^2 \leq e^{-2\delta t} V_s(0) + \frac{|\theta_1|^2}{4\delta} ||(\phi_1^s)'||^2_{L^2(0,1)} \tag{83}$$

which is similar to (22). We can also remark that (74), (75) and (76) are similar to (27), (28) and (44). Then we can propose the following result:

Theorem 4.1: *Consider system* (60a) *with initial condition* $w_0 \in H^2(0,1)$, $w(1) = 0 = w(0) = 0$, *and adaptive observer described by* (65). *Given* $\delta > 0$, *let* $N_0^s \in \mathbb{N}$ *satisfy* (66) *and* $N^s \in \mathbb{N}$ *satisfy* (81). *Let the vector of gains* $L_{N_0} = (l_1, \ldots, l_{N_0})^T$ *satisfy* (73). *Then under persistent excitation condition, the norms* $||\tilde{w}(.,t)||_{H^1(0,1)}$, $|\tilde{\theta}_1(t)|$ *and* $|\tilde{\theta}_2(t)|$ *converge to balls with a radius which is proportional to* $\frac{|\theta_1|}{\sqrt{\delta}} ||\phi_1^s||_{H^1(0,1)}$.

5. Example

In this section, we illustrate our result on system (1a). We consider $q = 3$ and $\phi_1(x,t) = \cos(\pi\ x/2) + \sin(t)\cos(\sqrt{3}\pi\ x/2)$. The output $y = u(0,t) + (2 - \cos(10t))\theta_2$. The condition (9) gives us $N_0 > \frac{1}{2} + \frac{\sqrt{3}}{\pi}$, then the smallest integer satisfying (9) is $N_0 = 2$. Given $\delta = 54$, then from (22), we have $N > 7/3$. From this, we deduce that the smallest integer satisfying (22) is $N = 3$. The simulations are performed with $N_0 = 2$ and $N = 3$, $L = (23.2, \quad 1.1)^T$. We also consider that $\theta_1 = 6.5$ and $\theta_2 = -2$. We also illustrate our result on system (60a) with $\nu = \pi^2/2$ and $\phi_1^s(x,t) = \sin(\pi x) + \cos(t)\sin(2\pi x)$ and $\delta = 54$. This conditions give us $N_0^s = 1$ and $N^s = 2$ (Figures 1 and 2).

6. Conclusion

In this paper, we presented new adaptive observers for heat and KSE equations. Our algorithms ensure good performances and are based only on a finite number of ODEs. The accuracy of our observers depends on the number of ODEs N. Further results concerning other classes of PDEs are under investigation.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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