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On stability of nonlinear homogeneous systems with distributed delays having variable kernels☆

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ABSTRACT

The stability problem for nonlinear homogeneous systems with distributed delay and variable kernel is studied. Both, the Lyapunov–Krasovskii and the Razumikhin, approaches are applied. It is proved that the global asymptotic stability of the zero solution for an auxiliary delay-free homogeneous system implies the local asymptotic stability of the zero solution for the original system with distributed delay. Moreover, the impact of nonlinear time-varying perturbations on the system dynamics is analyzed applying the averaging techniques. The results are illustrated by a mechanical system described by a Lienard equation, and an indirect control system design for a linear system.

1. Introduction

The problem of stability analysis for nonlinear systems is rather complex, and it becomes even more sophisticated being influenced by presence of time-delays and time-varying perturbations [1–5]. Raising the internet of things and cyber–physical systems technologies nowadays lead to appearance of scenarios, where all these factors meet together [6,7]. There are two main methods for stability analysis of time-delay systems: Lyapunov–Krasovskii (LK) and Lyapunov–Razumikhin (LR) approaches [5]. The former uses LK functionals (LKFs) and it has been proven to give the necessary and sufficient conditions of stability [8,9], while the latter is based on usual Lyapunov function analysis (under additional restrictions), and it also provides necessary and sufficient conditions of stability under mild restrictions [10]. The advantage of LR approach is its simplicity in application, since a Lyapunov function for the delay-free system can be tested as a guess candidate. Both methods have extensions to the input-to-state stability (ISS) verification for the systems with bounded disturbances [11,12].

Distributed delays can result from communication networks, the implementation of control/estimation algorithms [13–16] or human appearance in the loop [17]. Analyzing the stability of these systems requires specialized extensions of previously established methods [18, 19]. The complexity of the investigation increases when external perturbations are present, especially while assessing the permissible upper bounds of disturbances in relation to the delayed state (i.e., evaluating the asymptotic gains in terms of ISS). Considering the time-varying nature of the perturbations can lead to less conservative bounds. For instance, the efficiency of the averaging method has been demonstrated dealing with periodic or almost periodic perturbations [20,21].

When working in a nonlinear setting, it is advantageous to confine the analysis to a specific class of models. In this context, our focus will be on homogeneous dynamics, which have gained popularity due to their numerous beneficial properties in the absence of delays [22]. Furthermore, these properties have been extended to infinite-dimensional time-delay systems [23,24]. It has been recognized that for systems with discrete time-delays, if the delay-free counterparts (i.e., systems with zero delay) are globally asymptotically stable at the origin, then the original dynamics are locally asymptotically stable at the origin for any delay value, provided the positive homogeneity degree is present [23,25,26]. Alternatively, for systems with negative degree, the original dynamics are practically globally asymptotically stable, a property referred to as uniform ultimate boundedness of the solutions [27]. The LR method has been employed in the aforementioned works, and a recent development in this area is the LK approach, as presented in [28–31] (see [32] for the comparison of LR and LK approaches). Extension of these results to the case of distributed delays and exogenous suitably bounded perturbations admitting averaging was given

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in [16], where the distributed delays had constant kernels. Note that in the case of non-zero homogeneity degree (nonlinear setting), a wider class of perturbations (not just high frequency oscillations) preserves the stability compared to the linear counterparts, which is also the case of the present work.

In this paper, the stability problem for homogeneous systems of positive degree with distributed delay and variable kernel is studied. Both, the LK and the LR, approaches are applied. It is proven that the global asymptotic stability of the zero solution for an auxiliary delay-free homogeneous system implies the local asymptotic stability of the zero solution for the original system with distributed delay. Moreover, the impact of nonlinear time-varying perturbations on the system dynamics is analyzed. First, a theorem on the stability via nonlinear homogeneous approximation is proved. Next, with the aid of a special modification of the averaging approach, conditions of robust stability in the presence of nonlinear time-varying perturbations with zero mean values are derived. Compared with our recent work [16], the present contribution contains the following novelty:

(i) In [16], only distributed delays with constant kernels were considered, whereas here the case of variable kernels is investigated.

(ii) In the present paper, we propose special constructions of LR functions and LKFs that differ from those used in [16]. This permits us to obtain stability conditions for wider classes of perturbed systems.

On the other hand, in [16] both the cases, of positive and negative homogeneity degrees, were studied, whereas in this paper, we assume that the homogeneity degrees are only positive.

The outline of this work is as follows. Preliminaries are given in Section 2. The considered stability analysis problem is described in Section 3. The main results are formulated in Sections 4 and 5 with and without averaging tools, respectively. Examples of utilization of the proposed theoretical findings are shown in Section 6.

2. Preliminaries

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$, and $|s|$ is an absolute value for $s \in \mathbb{R}$. Euclidean norm for a real n-dimensional vector $x \in \mathbb{R}^n$ is defined as $\|x\|$. We denote by $C([-\tau, 0], \mathbb{R}^n)$, $0 < \tau < +\infty$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_\tau = \sup_{-\tau \leq \varsigma \leq 0} \|\phi(\varsigma)\|$.

For $\upsilon \in \mathbb{R}^n$, $\text{diag}\{\upsilon\}$ corresponds to a diagonal matrix with the components of vector υ on the main diagonal.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded.

The standard definitions of stability and related properties for time-delay systems can be found in [1,2,5], and for delay-free dynamics in [33].

2.1. Useful inequalities

The Young's inequality claims that for any $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ [34]:

$$\mathfrak{a}\mathfrak{b} \leq \frac{1}{p}\mathfrak{a}^p + \frac{p-1}{p}\mathfrak{b}^{\frac{p}{p-1}}$$

for any $p > 1$.

Using the properties of homogeneous functions the following results can be obtained:

Lemma 1 (*[29]*). *Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\ell > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ be given, then*

$$\mathfrak{a}^\alpha + \mathfrak{b}^\beta - \ell \mathfrak{a}^\gamma \mathfrak{b}^\delta \geq 0$$

provided that $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \leq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$ *and* $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$.

2.2. Homogeneity

For any $r_i > 0$, $i = \overline{1,n}$ and $\lambda > 0$, define the vector of weights $\mathbf{r} = [r_1, \ldots, r_n]$ and the dilation matrix

$$\Lambda_\mathbf{r}(\lambda) = \text{diag}\{(\lambda^{r_1}, \ldots, \lambda^{r_n})^\top\};$$

$$r_{\min} = \min_{i=\overline{1,n}} r_i \text{ and } r_{\max} = \max_{i=\overline{1,n}} r_i.$$

Definition 1 (*[22,35]*). The function $h : \mathbb{R}^n \to \mathbb{R}$ is called $\mathbf{r}$-homogeneous, if for any $x \in \mathbb{R}^n$ the relation

$$h(\Lambda_\mathbf{r}(\lambda)x) = \lambda^\nu h(x)$$

holds for some $\nu \in \mathbb{R}$ and all $\lambda > 0$.

The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathbf{r}$-homogeneous, if for any $x \in \mathbb{R}^n$ the relation

$$f(\Lambda_\mathbf{r}(\lambda)x) = \lambda^\nu \Lambda_\mathbf{r}(\lambda) f(x)$$

holds for some $\nu \geq -r_{\min}$ and all $\lambda > 0$.

In both cases, the constant ν is called the degree of homogeneity.

For any $x \in \mathbb{R}^n$ and $\varpi \geq r_{\max}$, a homogeneous norm can be defined as follows

$$|x|_r = \left(\sum_{i=1}^n |x_i|^{\varpi/r_i}\right)^{1/\varpi}.$$

For all $x \in \mathbb{R}^n$, its Euclidean norm $\|x\|$ is related with the homogeneous one:

$$\underline{\sigma}_r(|x|_r) \leq \|x\| \leq \bar{\sigma}_r(|x|_r)$$

for some $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$ [36]. In the following, due to this "equivalence", stability analysis with respect to the norm $\|x\|$ can be substituted with analysis for the norm $|x|_r$. The homogeneous norm has an important property: it is $\mathbf{r}$-homogeneous of degree 1, that is $|\Lambda_\mathbf{r}(\lambda)x|_r = \lambda |x|_r$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. Moreover, for any $\mathbf{r}$-homogeneous function $h : \mathbb{R}^n \to \mathbb{R}$ of degree $\nu \in \mathbb{R}$ there exist constants $\wp_1, \wp_2 \in \mathbb{R}_+$ such that

$$\wp_1 |x|_r^\nu \leq |h(x)| \leq \wp_2 |x|_r^\nu$$

for all $x \in \mathbb{R}^n$ [22].

3. Statement of the problem

Let a system with distributed delay

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^{t} D(s-t) G(x(s)) ds \tag{1}$$

be given, where $x(t) \in \mathbb{R}^n$, vector functions $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^q$ are continuous for $x \in \mathbb{R}^n$, the matrix function $D : [-\tau, 0] \to \mathbb{R}^{n\times q}$ is continuous, $\tau = \text{const} > 0$ is the maximum delay. Let initial functions for (1) belong to the space $C([-\tau, 0], \mathbb{R}^n)$, then the existence of continuous in time solutions follows [1, Theorem 2.1, p. 41]. Denote by x_t the restriction of a solution $x(t)$ to the segment $[t-\tau, t]$, i.e., $x_t : \xi \mapsto x(t+\xi)$, $\xi \in [-\tau, 0]$.

Assumption 1. Vector function $F(x)$ is $\mathbf{r}$-homogeneous of the degree $\nu > 0$ with respect to weights $\mathbf{r} = [r_1, \ldots, r_n]$, where $r_i > 0$, $i = 1, \ldots, n$:

$$F(\Lambda_\mathbf{r}(\lambda)x) = \lambda^\nu \Lambda_\mathbf{r}(\lambda) F(x), \ \forall x \in \mathbb{R}^n, \ \forall \lambda > 0.$$

Assumption 2. For every fixed $\theta \in [-\tau, 0]$, $D(\theta)G(x)$ is $\mathbf{r}$-homogeneous vector function of x of the degree $\nu > 0$:

$$D(\theta)G(\Lambda_\mathbf{r}(\lambda)x) = \lambda^\nu \Lambda_\mathbf{r}(\lambda) D(\theta) G(x), \ \forall x \in \mathbb{R}^n, \ \forall \lambda > 0.$$

Under these assumptions the system (1) admits the zero solution. We will look for conditions ensuring the (local) asymptotic stability of this solution. In addition, we will consider corresponding perturbed systems when exogenous perturbations are weighted by a state-dependent gain with distributed delay. With the aid of specially constructed LR functions or LKFs, and a modification of the averaging approach, we derive the conditions under which perturbations do not destroy the asymptotic stability.

Our standing assumption is stability of the delay-free counterpart of (1):

Assumption 3. The zero solution of the auxiliary homogeneous delay-free system

$$\dot{x}(t) = F(x(t)) + \int_{-\tau}^{0} D(\theta)d\theta\; G(x(t))$$

is asymptotically stable.

Remark 1. If Assumption 3 is satisfied, then (see [35,37]) there exists a Lyapunov function $V(x)$ with the following properties:

(i) $V(x)$ is twice continuously differentiable for $x \in \mathbb{R}^n$;

(ii) $V(x)$ is positive definite;

(iii) $V(x)$ is $\mathbf{r}$-homogeneous of the degree μ, where $\mu > r_i + r_j$, $i, j = 1, \dots, n$;

(iv) the function

$$\frac{\partial V(x)}{\partial x}\left(F(x) + \int_{-\tau}^{0} D(\theta)d\theta\; G(x)\right)$$

is negative definite.

Remark 2. In this work, the uniqueness of solutions for (1) is not required. Since the properties of a LKF or a LR function are verified for any map from $C([-\tau, 0], \mathbb{R}^n)$ being a solution of the system, in the case of existence of multiple solutions, our results will be obtained in the strong sense, i.e., for all solutions issues by an initial condition.

4. Stability analysis without averaging

First, let us formulate the conditions of stability for the nominal system (1).

Theorem 1. *Let Assumptions 1–3 be fulfilled. Then the zero solution of (1) is asymptotically stable.*

Proof. Construct a LKF candidate for (1) by the formula

$$\widetilde{V}(t, x_t) = V(x(t)) + \int_{t-\tau}^{t} (\alpha + \beta(s + \tau - t))|x(s)|_r^{\mu+\nu} ds + \frac{\partial V(x(t))}{\partial x}\int_{t-\tau}^{t}\int_{-\tau}^{s-t} D(\theta)d\theta G(x(s))ds, \quad (2)$$

where α, β are positive tuning parameters and $V(x)$ is a Lyapunov function with the properties specified in Remark 1.

Differentiating the functional (2) along the solutions of (1), we obtain

$$\dot{\widetilde{V}} = \frac{\partial V(x(t))}{\partial x}\left(F(x(t)) + \int_{-\tau}^{0} D(\theta)d\theta G(x(t))\right) + (\alpha + \beta\tau)|x(t)|_r^{\mu+\nu} - \alpha|x(t-\tau)|_r^{\mu+\nu}$$
$$-\beta\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds + \left(\int_{t-\tau}^{t}\int_{-\tau}^{s-t} D(\theta)d\theta G(x(s))ds\right)^{\top} \times\frac{\partial^2 V(x(t))}{\partial x^2}\left(F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds\right).$$

With the aid of properties of homogeneous functions, we arrive at the inequalities

$$c_1|x(t)|_r^{\mu} - c_3\sum_{i=1}^{n} |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i} ds + \alpha\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds \le \widetilde{V}(t, x_t) \le c_2|x(t)|_r^{\mu} + c_3\sum_{i=1}^{n} |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i} ds + (\alpha + \beta\tau)\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds,$$

$$\dot{\widetilde{V}} \le -c_4|x(t)|_r^{\mu+\nu} + c_5\sum_{i,j=1}^{n} |x(t)|_r^{\mu-r_i-r_j} \times\left(|x(t)|_r^{\nu+r_j} + \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_j} ds\right)\int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i} ds + (\alpha + \beta\tau)|x(t)|_r^{\mu+\nu} - \alpha|x(t-\tau)|_r^{\mu+\nu} - \beta\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds,$$

where c_1, c_2, c_3, c_4, c_5 are positive constants.

Choose the values of α and β such that $\alpha+\beta\tau < \frac{c_4}{4}$, then using Lemma 1, Young's and Hölder's inequalities, it is easy to verify that there exists a number $\delta > 0$ such that

$$\frac{1}{2}c_1|x(t)|_r^{\mu} + \frac{1}{2}\alpha\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds \le \widetilde{V}(t, x_t) \le 2c_2|x(t)|_r^{\mu} + 2(\alpha + \beta\tau)\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds,$$

$$\dot{\widetilde{V}} \le -\frac{1}{2}c_4|x(t)|_r^{\mu+\nu} - \frac{1}{2}\beta\int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds$$

for $\|x_t\|_\tau < \delta$. Hence (see [5]), the zero solution of (1) is asymptotically stable. The proof is completed. □

Next, along with (1), consider its perturbed version:

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds + \int_{t-\tau}^{t} Q(t, s, x(s))ds, \quad (3)$$

where the vector function

$$Q(t, s, x) = [Q_1(t, s, x), \dots, Q_n(t, s, x)]^{\top}$$

is continuous for $t \ge 0$, $s \in [t-\tau, t]$ and $x \in \mathbb{R}^n$.

Assumption 4. Let $|Q_i(t, s, x)| \le q_i|x|_r^{r_i+\sigma}$ for $t \ge 0$, $s \in [t-\tau, t]$, $x \in \mathbb{R}^n$, $i = 1, \dots, n$, where $q_i > 0$ and $\sigma > 0$.

This assumption is verified if for any fixed $t \ge 0$ and $s \in [t-\tau, t]$ the function $Q(t, s, x)$ is $\mathbf{r}$-homogeneous of x with the degree σ, and it is bounded for $t \ge 0$. We will look for conditions under which such generic time- and state-dependent perturbations do not destroy the asymptotic stability of the zero solution.

Theorem 2. *If Assumptions 1–4 are fulfilled and $\sigma > \nu$, then the zero solution of (3) is asymptotically stable.*

Proof. Differentiating the functional (2) along the solutions of (3), we obtain

$$\dot{\tilde{V}} = W(t,x_t) + \frac{\partial V(x(t))}{\partial x}\int_{t-\tau}^{t} Q(t,s,x(s))ds + \left(\int_{t-\tau}^{t}\int_{-\tau}^{s-t} D(\theta)d\theta G(x(s))ds\right)^{\top} \times \frac{\partial^2 V(x(t))}{\partial x^2}\int_{t-\tau}^{t} Q(t,s,x(s))ds$$
$$\le W(t,x_t) + a_1\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds$$
$$+a_2\sum_{i,j=1}^{n}|x(t)|_r^{\mu-r_i-r_j}\int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_j}ds,$$

where a_1, a_2 are positive constants and $W(t,x_t)$ is the derivative of (2) along the solutions of the system (1).

The remaining part of the proof is similar to that of Theorem 1. □

Remark 3. Under the conditions of Theorem 2, the system (1) can be interpreted as a nonlinear approximation at the origin for (3) [22]. The theorem guarantees the preservation of the asymptotic stability if degrees of perturbations are higher than those of the right-hand sides of the original system.

Remark 4. The result of Theorem 2 can be interpreted as local ISS property of (1) with respect to additive essentially bounded disturbances $d(t) \in \mathbb{R}^n$:

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds + d(t),$$

then the term $\int_{t-\tau}^{t} Q(t,s,x(s))ds$ satisfying the conditions of Assumption 4 corresponds to an upper bound on the asymptotic gain of the system.

In the next section, we will consider systems with time-varying perturbations possessing zero mean values. On the basis of a development of the averaging method, we will show that in such a case the asymptotic stability can be guaranteed under less conservative constraints than those formulated in Theorem 2.

5. Stability analysis via averaging

We will consider two types of perturbed systems with different dependence on external disturbances. It is worth noticing that, for the first type, to derive stability conditions, the LK approach is used, while we failed to prove a similar result with the aid of the LR approach. For the second type, the situation is opposite.

5.1. Application of the LK approach

Let the system (3) be of the form

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds + \int_{t-\tau}^{t} B(s)L(s-t)Q(x(s))ds, \tag{4}$$

where the matrix function $B : [-\tau, +\infty) \to \mathbb{R}^{n\times w}$ is continuous and bounded, the matrix function $L : [-\tau, 0] \to \mathbb{R}^{w\times q}$ is continuous, the vector function $Q : \mathbb{R}^n \to \mathbb{R}^q$ is continuously differentiable, and the remaining notation is the same as for (1).

Assumption 5. For any fixed $t \ge 0$ and $s \in [t-\tau, t]$, $B(s)L(s-t)Q(x)$ is $\mathbf{r}$-homogeneous function of x with the degree $\sigma > 0$.

This assumption is a special case of Assumption 4 for the system (4).

Moreover, in this section, it is supposed that time-varying perturbations admit zero mean values. We will consider such a constraint in one of the following forms.

Assumption 6. Let

$$\left\|\int_0^t B(s)ds\right\| \le N \quad \text{for} \quad t \ge 0, \quad N = \text{const} > 0.$$

Assumption 7. Let

$$\frac{1}{T}\int_t^{t+T} B(s)ds \to 0 \quad \text{as} \quad T \to +\infty$$

uniformly with respect to $t \ge 0$.

In particular, Assumption 6 is fulfilled when entries of $B(t)$ are periodic functions with zero mean values. Assumption 7 is valid, e.g., in the case where entries of $B(t)$ are almost periodic functions with zero mean values. It is worth noticing that matrices with such entries may not satisfy Assumption 6 (see [38]).

Theorem 3. *Let Assumptions 1–3, 5 be fulfilled. Then the zero solution of (4) is asymptotically stable*
(i) under Assumption 6 for $\sigma > \nu/2$;
(ii) under Assumption 7 for $\sigma \ge \nu$.

Proof. First, choose a LKF candidate for (4) as follows:

$$\tilde{V}_1(t,x_t) = \frac{\partial V(x(t))}{\partial x}\int_{t-\tau}^{t} B(s)\int_{-\tau}^{s-t} L(\theta)d\theta Q(x(s))ds + \tilde{V}(t,x_t),$$

where $\tilde{V}(t,x_t)$ is the functional constructed by the formula (2) and the function $V(x)$ possesses the properties specified in Remark 1. Differentiating $\tilde{V}_1(t,x_t)$ along the solutions of (4), we obtain

$$\dot{\tilde{V}}_1 = \frac{\partial V(x(t))}{\partial x}\left(F(x(t)) + \int_{-\tau}^{0} D(\theta)d\theta G(x(t))\right) + \frac{\partial V(x(t))}{\partial x}B(t)\int_{-\tau}^{0} L(\theta)d\theta\, Q(x(t))$$
$$+(\alpha+\beta\tau)|x(t)|_r^{\mu+\nu} - \alpha|x(t-\tau)|_r^{\mu+\nu} - \beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds$$
$$+\Big(F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds + \int_{t-\tau}^{t} B(s)L(s-t)Q(x(s))ds\Big)^{\top}\left(\frac{\partial^2 V(x(t))}{\partial x^2}\right)^{\top} \times\Big(\int_{t-\tau}^{t}\int_{-\tau}^{s-t} D(\theta)d\theta G(x(s))ds + \int_{t-\tau}^{t} B(s)\int_{-\tau}^{s-t} L(\theta)d\theta Q(x(s))ds\Big).$$

Applying the properties of homogeneous functions (see [22]), we arrive at the estimate

$$\dot{\tilde{V}}_1 \le (\alpha+\beta\tau-a_1)|x(t)|_r^{\mu+\nu} + \frac{\partial V(x(t))}{\partial x}B(t)\int_{-\tau}^{0} L(\theta)d\theta\, Q(x(t))$$
$$-\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds + a_2\sum_{i,j=1}^{n}|x(t)|_r^{\mu-r_i-r_j} \times\left(|x(t)|_r^{\nu+r_j} + \varrho_j(x_t)\right)\varrho_i(x_t),$$

where a_1, a_2 are positive coefficients and $\varrho_i(x_t) = \int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_i} + |x(s)|_r^{\sigma+r_i}\right)ds$. Next, according to the approach developed in [31,39] construct a LKF in the form

$$\tilde{V}_2(t,x_t) = \tilde{V}_1(t,x_t) - \frac{\partial V(x(t))}{\partial x} \times\int_0^t e^{\varepsilon(s-t)}B(s)ds\int_{-\tau}^{0} L(\theta)d\theta\, Q(x(t)),$$

where ε is a nonnegative tuning parameter, then

$$\begin{aligned}
&c_1|x(t)|_r^{\mu} - c_3\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\varrho_i(x_t)\\
&-c_4\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma} + \alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds\\
&\le \widetilde{V}_2(t,x_t) \le c_2|x(t)|_r^{\mu} + c_3\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\varrho_i(x_t)\\
&+c_4\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma}\\
&+(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,\\
&\dot{\widetilde{V}}_2 \le (\alpha+\beta\tau-a_1)|x(t)|_r^{\mu+\nu}\\
&+a_3\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma} - \beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds\\
&+a_2\sum_{i,j=1}^{n}|x(t)|_r^{\mu-r_i-r_j}\left(|x(t)|_r^{\nu+r_j}+\varrho_j(x_t)\right)\varrho_i(x_t)\\
&+a_4\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\|\sum_{j=1}^{n}|x(t)|_r^{\mu+\sigma-r_j}\\
&\times\left(|x(t)|_r^{\nu+r_j}+\varrho_j(x_t)\right),
\end{aligned}$$

where $c_1, c_2, c_3, c_4, a_3, a_4$ are positive constants. With the aid of Lemma 1, Young's and Hölder's inequalities, it can be shown that if $\alpha+\beta\tau < a_1/2$, $\sigma > \nu/2$ and a number $\delta > 0$ is sufficiently small, then the inequalities

$$\begin{aligned}
&\frac{1}{2}c_1|x(t)|_r^{\mu} - c_4\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma}\\
&+\frac{1}{2}\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \le \widetilde{V}_2(t,x_t)\\
&\le 2c_2|x(t)|_r^{\mu} + c_4\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma}\\
&+2(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,
\end{aligned}$$

$$\begin{aligned}
&\dot{\widetilde{V}}_2 \le -\frac{1}{2}a_1|x(t)|_r^{\mu+\nu} + a_3\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma}\\
&-\frac{1}{2}\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds + a_4\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\|\\
&\times\sum_{j=1}^{n}|x(t)|_r^{\mu+\sigma-r_j}\left(|x(t)|_r^{\nu+r_j}+\varrho_j(x_t)\right)
\end{aligned}$$

hold for $t \ge 0$, $\|x_t\|_\tau < \delta$. Next, consider the cases of Assumptions 6 and 7 separately.

(1) Let Assumption 6 be fulfilled. In this case one can take $\varepsilon = 0$. We obtain

$$\begin{aligned}
&\frac{1}{2}c_1|x(t)|_r^{\mu} - c_4N|x(t)|_r^{\mu+\sigma} + \frac{1}{2}\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \le \widetilde{V}_2(t,x_t)\\
&\le 2c_2|x(t)|_r^{\mu} + c_4N|x(t)|_r^{\mu+\sigma} + 2(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,
\end{aligned}$$

$$\begin{aligned}
&\dot{\widetilde{V}}_2 \le -\frac{1}{2}a_1|x(t)|_r^{\mu+\nu} - \frac{1}{2}\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds\\
&+a_4N\sum_{j=1}^{n}|x(t)|_r^{\mu+\sigma-r_j}\\
&\times\left(|x(t)|_r^{\nu+r_j} + \int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_j}+|x(s)|_r^{\sigma+r_j}\right)ds\right)
\end{aligned}$$

for $t \ge 0$, $\|x_t\|_\tau < \delta$. Hence, if $\sigma > \nu/2$ and δ is sufficiently small, then the estimates

$$\begin{aligned}
&\frac{1}{3}c_1|x(t)|_r^{\mu} + \frac{1}{3}\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \le \widetilde{V}_2(t,x_t)\\
&\le 3c_2|x(t)|_r^{\mu} + 3(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,
\end{aligned} \tag{5}$$

$$\dot{\widetilde{V}}_2 \le -\frac{1}{3}a_1|x(t)|_r^{\mu+\nu} - \frac{1}{3}\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \tag{6}$$

are valid for $t \ge 0$ and $\|x_t\|_\tau < \delta$.

(2) Let Assumption 7 be fulfilled. In this case ε should be positive. We obtain

$$\begin{aligned}
&\frac{1}{2}c_1|x(t)|_r^{\mu} - \frac{c_5}{\varepsilon}|x(t)|_r^{\mu+\sigma} + \frac{1}{2}\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \le \widetilde{V}_2(t,x_t)\\
&\le 2c_2|x(t)|_r^{\mu} + \frac{c_5}{\varepsilon}|x(t)|_r^{\mu+\sigma} + 2(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,
\end{aligned}$$

$$\begin{aligned}
&\dot{\widetilde{V}}_2 \le -\frac{1}{2}a_1|x(t)|_r^{\mu+\nu} + a_3\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x(t)|_r^{\mu+\sigma}\\
&-\frac{1}{2}\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds + \frac{a_5}{\varepsilon}\sum_{i,j=1}^{n}|x(t)|_r^{\mu+\sigma-r_j}\\
&\times\left(|x(t)|_r^{\nu+r_j} + \int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_j}+|x(s)|_r^{\sigma+r_j}\right)ds\right)
\end{aligned}$$

for $t \ge 0$, $\|x_t\|_\tau < \delta$, where a_5 and c_5 are positive constants. Let $\sigma \ge \nu$, in [20] it was proven that, under Assumption 7, $\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\| \to 0$ as $\varepsilon \to 0$ uniformly with respect to $t \ge 0$. Choose $\varepsilon > 0$ satisfying the condition

$$\varepsilon a_3\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\| < \frac{a_1}{12}$$

for $t \ge 0$. Then there exists $\delta > 0$ such that (5) and (6) hold for $t \ge 0$ and $\|x_t\|_\tau < \delta$.

This completes the proof. □

Remark 5. The obtained LKF V_2 slightly differs from the one used in [31], where also stronger regularity requirements are imposed on F, G and Q.

5.2. Application of the LR approach

Consider the perturbed system (3) of the form

$$\begin{aligned}
\dot{x}(t) = F(x(t)) &+ \int_{t-\tau}^{t}D(s-t)G(x(s))ds\\
&+ B(t)\int_{t-\tau}^{t}L(s-t)Q(x(s))ds,
\end{aligned} \tag{7}$$

where the notation is the same as for (1) and (4).

Assumption 8. For any fixed $t \ge 0$ and $s \in [t-\tau, t]$, $B(t)L(s-t)Q(x)$ is $\mathbf{r}$-homogeneous function of x with the degree $\sigma > 0$.

Again, it is a particular case of Assumption 4 for the system (7).

Theorem 4. *Let Assumptions 1–3, 8 be fulfilled. Then the zero solution of (7) is asymptotically stable*

(i) under Assumption 6 for $\sigma > \nu/2$;
(ii) under Assumption 7 for $\sigma \ge \nu$.

Proof. Following [25], choose a Lyapunov function candidate for (7) in the form

$$\begin{aligned}
V_1(t,x) = V(x) &- \frac{\partial V(x)}{\partial x}\int_0^t e^{\varepsilon(s-t)}B(s)ds\\
&\times\int_{-\tau}^{0}L(\theta)d\theta\, Q(x),
\end{aligned}$$

where ε is a nonnegative tuning parameter and $V(x)$ is a Lyapunov function with the properties specified in Remark 1. Then

$$\begin{aligned}
&c_1|x|_r^{\mu} - c_3\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x|_r^{\mu+\sigma} \le V_1(t,x)\\
&\le c_2|x|_r^{\mu} + c_3\left\|\int_0^t e^{\varepsilon(s-t)}B(s)ds\right\||x|_r^{\mu+\sigma},
\end{aligned}$$

$$\dot{V}_1 = \frac{\partial V(x(t))}{\partial x}\left(F(x(t)) + \int_{-\tau}^{0} D(\theta)d\theta\, G(x(t))\right)$$
$$+\varepsilon \frac{\partial V(x(t))}{\partial x}\int_0^t e^{\varepsilon(s-t)}B(s)ds \int_{-\tau}^{0} L(\theta)d\theta\, Q(x(t))$$
$$+\frac{\partial V(x(t))}{\partial x}\int_{t-\tau}^{t} D(s-t)(G(x(s))-G(x(t)))ds$$
$$+\frac{\partial V(x(t))}{\partial x}B(t)\int_{t-\tau}^{t} L(s-t)(Q(x(s))-Q(x(t)))ds$$
$$-\frac{\partial V(x(t))}{\partial x}\int_0^t e^{\varepsilon(s-t)}B(s)\,ds \int_{-\tau}^{0} L(\theta)d\theta\, \frac{\partial Q(x(t))}{\partial x}$$
$$\times[F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds$$
$$+B(t)\int_{t-\tau}^{t} L(s-t)Q(x(s))ds]$$
$$-[F(x(t)) + \int_{t-\tau}^{t} D(s-t)G(x(s))ds$$
$$+B(t)\int_{t-\tau}^{t} L(s-t)Q(x(s))ds]^{\top}$$
$$\times\frac{\partial^2 V(x(t))}{\partial x^2}\int_0^t e^{\varepsilon(s-t)}B(s)\,ds \int_{-\tau}^{0} L(\theta)d\theta\; Q(x(t))$$
$$\le -c_4|x(t)|_r^{\mu+\nu} + c_5\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)\,ds\right\| |x(t)|_r^{\mu+\sigma}$$
$$+c_6\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |R_i(s-t,x(s)) - R_i(s-t,x(t))|ds$$
$$+c_7\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |H_i(t,s,x(s)) - H_i(t,s,x(t))|ds$$
$$+c_8\left\|\int_0^t e^{\varepsilon(s-t)}B(s)\,ds\right\|\sum_{i=1}^n |x(t)|_r^{\mu+\sigma-r_i}[|x(t)|_r^{\nu+r_i}$$
$$+\int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i}ds + \int_{t-\tau}^{t} |x(s)|_r^{\sigma+r_i}ds],$$

where $c_k > 0$, $k = 1,\dots,8$, $H_i(t,s,x)$ are components of the vector function $B(t)L(s-t)Q(x)$ and $R_i(s-t,x)$ are components of the vector function $D(s-t)G(x)$.

Consider again separately the cases of Assumptions 6 and 7.

(1) Let Assumption 6 be fulfilled. In this case one can take $\varepsilon = 0$. We obtain

$$c_1|x|_r^{\mu} - c_3N|x|_r^{\mu+\sigma} \le V_1(t,x) \le c_2|x|_r^{\mu} + c_3N|x|_r^{\mu+\sigma},$$

$$\dot{V}_1 \le -c_4|x(t)|_r^{\mu+\nu}$$
$$+c_6\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |R_i(s-t,x(s)) - R_i(s-t,x(t))|ds$$
$$+c_7\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |H_i(t,s,x(s)) - H_i(t,s,x(t))|ds$$
$$+c_8N\sum_{i=1}^n |x(t)|_r^{\mu+\sigma-r_i}[|x(t)|_r^{\nu+r_i} + \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i}ds$$
$$+\int_{t-\tau}^{t} |x(s)|_r^{\sigma+r_i}ds].$$

Choose a number $\delta > 0$ such that

$$\frac{1}{2}c_1|x|_r^{\mu} \le V_1(t,x) \le 2c_2|x|_r^{\mu}$$

for $t \ge 0$, $|x|_r < \delta$. Assume that $0 < |x(\xi)|_r < \delta$ for $\xi \in [t-2\tau,t]$ and the function $V_1(t,x)$ satisfies the Razumikhin condition $V_1(\xi,x(\xi)) \le 2V_1(t,x(t))$ for $\xi \in [t-2\tau,t]$ (we need to enlarge the delay value due to technical reasons), then

$$|x(\xi)|_r \le \omega|x(t)|_r,\ \forall\xi \in [t-2\tau,t] \tag{8}$$

for some $\omega > 1$ and

$$\int_{t-\tau}^{t} |R_i(s-t,x(s)) - R_i(s-t,x(t))|ds$$
$$= |x(t)|_r^{\nu+r_i}\int_{t-\tau}^{t} |R_i(s-t,z(t)+\Delta z(t,s)) - R_i(s-t,z(t))|ds,$$

where $z(t) = \Lambda_{\mathbf{r}}^{-1}(|x(t)|_r)x(t)$, $\Delta z(t,s) = \Lambda_{\mathbf{r}}^{-1}(|x(t)|_r)(x(s)-x(t))$, $i = 1,\dots,n$. Using Mean Value Theorem, it is easy to verify that there exists a number $\tilde{m} > 0$ such that

$$\|\Delta z(t,s)\| \le \tilde{m}(|x(t)|_r^{\nu} + |x(t)|_r^{\sigma}).$$

Hence,

$$\int_{t-\tau}^{t} |R_i(s-t,z(t)+\Delta z(t,s)) - R_i(s-t,z(t))|ds \to 0$$

as $|x(t)|_r \to 0$. Next,

$$\int_{t-\tau}^{t} |H_i(t,s,x(s)) - H_i(t,s,x(t))|ds$$
$$= \int_{t-\tau}^{t}\left|(s-t)\sum_{j=1}^n \frac{\partial H_i(t,s,x(\theta_i(s,t)))}{\partial x_j}\dot{x}_j(\theta_i(s,t))\right|ds$$
$$\le c_9\tau^2\sum_{j=1}^n |x(t)|_r^{\sigma+r_i-r_j}\left(|x(t)|_r^{\nu+r_j} + |x(t)|_r^{\sigma+r_j}\right)ds,$$

where we used the Mean Value Theorem on the first step for some $\theta_i(s,t) \in [t-\tau,t]$, and (8) to get the final estimate for a constant $c_9 > 0$. Taking into account this result and the estimate (8), applying Young's inequality and Lemma 1, it can be proved that if $\sigma > \nu/2$ and the value of δ is sufficiently small, then

$$\dot{V}_1 \le -\frac{1}{2}c_4|x(t)|_r^{\mu+\nu}.$$

(2) Let Assumption 7 be fulfilled. In this case ε should be positive. We obtain

$$c_1|x|_r^{\mu} - \frac{c_{11}}{\varepsilon}|x|_r^{\mu+\sigma} \le V_1(t,x) \le c_2|x|_r^{\mu} + \frac{c_{11}}{\varepsilon}|x|_r^{\mu+\sigma},$$

$$\dot{V}_1 \le -c_4|x(t)|_r^{\mu+\nu} + c_5\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)\,ds\right\| |x(t)|_r^{\mu+\sigma}$$
$$+c_6\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |R_i(s-t,x(s)) - R_i(s-t,x(t))|ds$$
$$+c_7\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |H_i(t,s,x(s)) - H_i(t,s,x(t))|ds$$
$$+\frac{c_{10}}{\varepsilon}\sum_{i=1}^n |x(t)|_r^{\mu+\sigma-r_i}[|x(t)|_r^{\nu+r_i} + \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i}ds$$
$$+\int_{t-\tau}^{t} |x(s)|_r^{\sigma+r_i}ds],$$

where c_{10} and c_{11} are positive constants. Let $\sigma \ge \nu$ and choose $\varepsilon > 0$ such that

$$c_5\varepsilon\left\|\int_0^t e^{\varepsilon(s-t)}B(s)\,ds\right\| < \frac{c_4}{2}$$

for $t \ge 0$, then

$$\dot{V}_1 \le -c_4\left(1 - \frac{1}{2}|x(t)|_r^{\sigma-\nu}\right)|x(t)|_r^{\mu+\nu}$$
$$+c_6\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |R_i(s-t,x(s)) - R_i(s-t,x(t))|ds$$
$$+c_7\sum_{i=1}^n |x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t} |H_i(t,s,x(s)) - H_i(t,s,x(t))|ds$$
$$+\frac{c_{10}}{\varepsilon}\sum_{i=1}^n |x(t)|_r^{\mu+\sigma-r_i}[|x(t)|_r^{\nu+r_i} + \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i}ds$$
$$+\int_{t-\tau}^{t} |x(s)|_r^{\sigma+r_i}ds].$$

In this case we can use the representation

$$\int_{t-\tau}^{t} |H_i(t,s,x(s)) - H_i(t,s,x(t))|ds$$
$$= |x(t)|_r^{\sigma+r_i} \int_{t-\tau}^{t} |H_i(t,s,z(t)+\Delta z(t,s)) - H_i(t,s,z(t))|ds.$$

The subsequent proof is similar to that one for the previous case. □

Remark 6. The results of Theorems 3 and 4 improve the condition of Theorem 2 that the degree of the perturbations σ should be strictly bigger than the degree of the nominal system ν. Under Assumptions 6 and 7, which allow the averaging tools to be applied in the proof, σ can be chosen at least equal to ν or even higher than $\nu/2$.

6. Applications

In this section the results of Theorems 3 and 4 will be adapted to two practical cases.

6.1. Vector Lienard equation

Assume that motions of a mechanical system are modeled by the equations

$$\ddot{x}(t) + \frac{\partial W(x(t))}{\partial x}\dot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x}^{\top} + \int_{t-\tau}^{t} d(s-t)\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}^{\top} ds + \int_{t-\tau}^{t} B(s)L(s-t)Q(x(s))ds = 0, \tag{9}$$

where $x(t) \in \mathbb{R}^n$; components of the vector functions $W(x)$ and $Q(x)$ are continuously differentiable for $x \in \mathbb{R}^n$ standard homogeneous (with $\mathbf{r} = [1,\dots,1]$) of the degree $\nu+1$ and λ, respectively, $\nu > 0$, $\lambda > 1$; scalar functions $\Pi(x)$ and $\widetilde{\Pi}(x)$ are continuously differentiable for $x \in \mathbb{R}^n$ and standard homogeneous of the degree $\rho+1$, $\rho > 1$; the matrix $B(t)$ is continuous for $t \in [-\tau,+\infty)$, the scalar function $d(\theta)$ and the matrix $L(\theta)$ are continuous for $\theta \in [-\tau, 0]$.

In the case where $\tau = 0$ the system (9) is a classical vector type Lienard equation describing the dynamics of various mechanical and electromechanical systems (see [40]). The term $\int_{t-\tau}^{t} d(s-t)\partial\widetilde{\Pi}(x(s))/\partial x^{\top} ds$ can be interpreted as integral part of a PID regulator [13,41–43], whereas the term $\int_{t-\tau}^{t} B(s)L(s-t)Q(x(s))ds$ characterizes external time-varying perturbations acting on the system (e.g., through the control channel).

Using the substitution $y(t) = \dot{x}(t) + W(x(t))$, transform (9) into a first-order system:

$$\dot{x}(t) = y(t) - W(x(t)),$$
$$\dot{y}(t) = -\frac{\partial \Pi(x(t))}{\partial x}^{\top} - \int_{t-\tau}^{t} d(s-t)\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}^{\top} ds - \int_{t-\tau}^{t} B(s)L(s-t)Q(x(s))ds.$$

Let $\rho = 2\nu+1$, $\lambda > \nu+1$, then the corresponding nominal dynamics

$$\dot{x}(t) = y(t) - W(x(t)),$$
$$\dot{y}(t) = -\frac{\partial \Pi(x(t))}{\partial x}^{\top} - \int_{t-\tau}^{t} d(s-t)\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}^{\top} ds$$

is $\mathbf{r}$-homogeneous of the degree ν with respect to the dilation $\mathbf{r} = [r_1,\dots,r_{2n}]$, where $r_i = 1$ for $i = 1,\dots,n$ and $r_i = \nu+1$ for $i = n+1,\dots,2n$, and the function $B(t)L(s-t)Q(x)$ satisfies Assumption 5 with $\sigma = \lambda-\nu-1$.

Furthermore, let the functions $\Pi(x) + \int_{-\tau}^{0} d(\theta)d\theta\,\widetilde{\Pi}(x)$ and

$$\left(\frac{\partial \Pi(x)}{\partial x} + \int_{-\tau}^{0} d(\theta)d\theta\,\frac{\partial \widetilde{\Pi}(x)}{\partial x}\right)W(x)$$

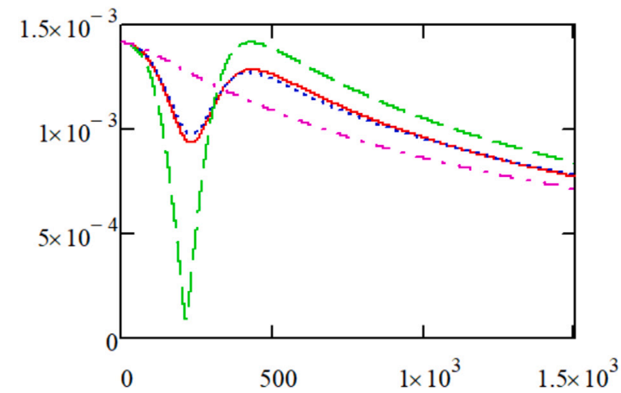


Fig. 1. Behavior of $|x(t)| + |\dot{x}(t)|$ in logarithmic scale, $t \in [0, 1500]$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

be positive definite, then it is known (see [44]) that, under this condition, the zero solution of the auxiliary delay-free system

$$\dot{x}(t) = y(t) - W(x(t)),$$
$$\dot{y}(t) = -\frac{\partial \Pi(x(t))}{\partial x}^{\top} - \int_{-\tau}^{0} d(\theta)d\theta\,\frac{\partial \widetilde{\Pi}(x(t))}{\partial x}^{\top}$$

is asymptotically stable.

Applying Theorem 3, we obtain that if $\lambda > 1 + \frac{3}{2}\nu$ and Assumption 6 is fulfilled, then the equilibrium position $x = \dot{x} = 0$ of (9) is asymptotically stable.

Example 1. Let

$$W(x) = \begin{bmatrix} 8x_1^{\nu}|x_1| + 0.5x_2^{\nu+1} \\ x_1^{\nu+1} + 8x_2^{\nu}|x_2| \end{bmatrix},\quad B(t) = \cos(\frac{\pi}{\tau}t),$$
$$d(t) = \sin(\frac{2\pi}{\tau}t),\quad 3\Pi(x) = \widetilde{\Pi}(x) = x_1^{\rho+1} + x_2^{\rho+1},$$
$$Q(x) = \begin{bmatrix} -2x_1^{\lambda} - 3x_1^{\lambda-1}x_2 + 3x_2^{\lambda} \\ 3x_1^{\lambda} + 3x_1x_2^{\lambda-1} - 2x_2^{\lambda} \end{bmatrix},\quad L(t) = \sin^2(t),$$
$$\nu = 1,\ \lambda = \rho = 2\nu+1,\ \tau = 0.1,$$

which verify the conditions established above. The results of simulation of the closed-loop system (the explicit Euler method was used with step $\Delta t = 0.01$) for different initial conditions with the same norm (they were chosen constant for $s \in [-\tau, 0]$ and taking random values) are shown in Fig. 1, where distinct colors correspond to different initial conditions.

6.2. A system of indirect control

It should be noted that the approaches developed in this paper can also be applied to some classes of nonlinear systems that are not homogeneous. As an example of such a case, consider an indirect control system [45] of the form

$$\dot{x}(t) = Ax(t) + b(t)u(t),\ \dot{z}(t) = c^{\top}x(t) + h(t)u(t),$$

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, $c \in \mathbb{R}^n$ is a constant vector, vector $b(t) \in \mathbb{R}^n$ and scalar $h(t)$ functions are continuous and bounded for $t \in (-\infty,+\infty)$, $u(t)$ is a scalar control.

Assume that $b(t) = \bar{b} + \tilde{b}(t)$, $h(t) = \bar{h} + \tilde{h}(t)$, where $\bar{h}$ is a constant, $\bar{b} \in \mathbb{R}^n$ is a constant vector, whereas

$$\frac{1}{T}\int_{t}^{t+T} \tilde{b}(s)ds \to 0,\quad \frac{1}{T}\int_{t}^{t+T} \tilde{h}(s)ds \to 0 \ \text{ as }\ T \to +\infty$$

uniformly with respect to $t \geq 0$ (Assumption 7 is verified). Hence, the corresponding averaged system takes the form

$$\dot{x}(t) = Ax(t) + \bar{b}u(t),\ \dot{z}(t) = c^{\top}x(t) + \bar{h}u(t).$$

For this nominal model, choose a control law as follows:

$$u(t) = pz^{\nu}(t) + \int_{t-\tau}^{t} d(s-t)z^{\nu}(s)ds,$$

where scalar function $d(\theta)$ is continuous for $\theta \in [-\tau, 0]$, $p \in \mathbb{R}$ determines the delay-independent part, $\nu > 1$ is a rational with odd numerator and denominator, τ is a constant positive delay. Appearance of a distributed delay is related with control communication media, then the nominal feedback is proportional to $z^{\nu}(t)$, as it can be observed in a delay-free counterpart of the controlled system:

$$\dot{x}(t) = Ax(t) + \left(p + \int_{-\tau}^{0} d(\theta)d\theta\right) z^{\nu}(t)\bar{b}, \tag{10}$$

$$\dot{z}(t) = c^{\top}x(t) + \bar{h}\left(p + \int_{-\tau}^{0} d(\theta)d\theta\right) z^{\nu}(t).$$

Note also that the distributed delay can be introduced artificially to improve the transients [46]. According to the standard assumptions (see [45]), consider the case where A is a Hurwitz matrix and $\left(p + \int_{-\tau}^{0} d(\theta)d\theta\right)(\bar{h} - c^{\top}A^{-1}\bar{b}) < 0$, then the system (10) admits a strict Lyapunov function of the form

$$V(x,z) = \frac{z^{\nu+1}}{\nu+1} + x^{\top}Px - z^{\nu}c^{\top}A^{-1}x$$

with a constant positive definite matrix P.

Using this function, in a similar way as in the proof of Theorem 4 it can be shown that the zero solution of the perturbed closed-loop system

$$\dot{x}(t) = Ax(t) + b(t)\left(pz^{\nu}(t) + \int_{t-\tau}^{t} d(s-t)z^{\nu}(s)ds\right),$$

$$\dot{z}(t) = c^{\top}x(t) + h(t)\left(pz^{\nu}(t) + \int_{t-\tau}^{t} d(s-t)z^{\nu}(s)ds\right)$$

is asymptotically stable.

Example 2. Let $n = 1$,

$$A = -1, \bar{b} = -2, c = \bar{h} = 1, p = 1,$$

then for $\nu = 1$ (the case is not studied in the paper) and $\tau = 0$ the system (10) takes the form:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}x(t)\\ z(t)\end{bmatrix},$$

and it has purely imaginary eigenvalues. However, since all conditions imposed in this work are satisfied, for $\nu > 1$ the system will be locally asymptotically stable.

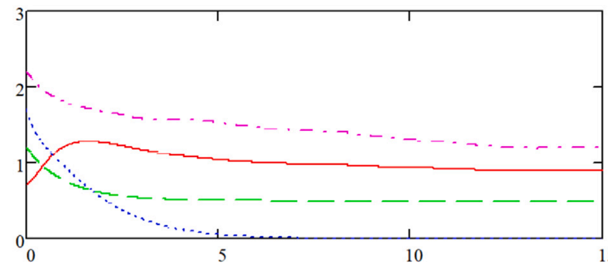
Example 3. Let

$$A = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix},\ c = \begin{bmatrix}1\\ 1\end{bmatrix},\ \bar{b} = \begin{bmatrix}1\\ -1\end{bmatrix},$$

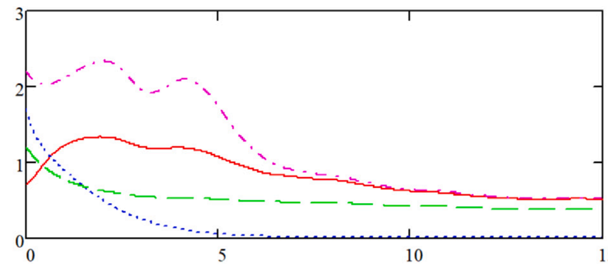
$$\bar{h} = -1,\ \tilde{b}(t) = \begin{bmatrix}\sin((1+\cos^2(t))t)\\ \cos(2t)\end{bmatrix},\ \tilde{h}(t) = \sin(0.5t);$$

$$p = 0.01,\ d(s) = \exp(2s),\ s \in [-\tau, 0];\ \nu = 3,$$

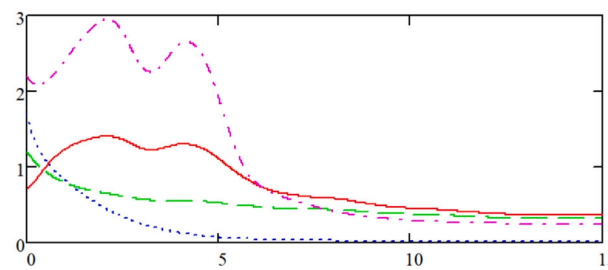
then all required conditions are satisfied. The results of simulation of the controlled system for different initial conditions (chosen constant for $s \in [-\tau, 0]$) are shown in Fig. 2 for values of delay $\tau \in \{0, 0.1, 0.25\}$ (the explicit Euler method was used with step $\Delta t = 0.01$). As we can conclude, in the delay-free case the system is slowly converging with the control $u(t) = pz^{\nu}(t)$, for a small value of delay we observe an accelerated convergence, while increasing the value of τ leads to more complex transients with a better convergence.



(a) $\tau = 0$



(b) $\tau = 0.1$



(c) $\tau = 0.25$

Fig. 2. Behavior of $|x(t)| + |z(t)|$ for different initial conditions and delays τ, $t \in [0, 15]$.

7. Conclusion

The stability problem for nonlinear homogeneous systems with distributed delay featuring a variable kernel has been investigated. Both the LK and LR approaches have been utilized. It has been demonstrated that the global asymptotic stability of the zero solution for an auxiliary delay-free homogeneous system implies the local asymptotic stability of the zero solution for the original dynamics with distributed delay. Additionally, the impact of nonlinear time-varying perturbations on the system's behavior has been analyzed using averaging techniques. The obtained bounds on admissible homogeneity degrees of the perturbation terms coincide with the respective analogues derived for delay-free systems [47] or systems with constant delays [25,31], hence, can be considered as rather accurate. These findings have been exemplified through a mechanical system described by a Lienard equation, as well as an indirect control system design for a linear system. Future directions of research may include extensions of these results to the case of the negative homogeneity degree.

CRediT authorship contribution statement

A. Aleksandrov: Writing – original draft, Supervision, Conceptualization. **D. Efimov:** Investigation. **E. Fridman:** Writing – original draft, Supervision, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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