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Finite-Dimensional Adaptive Observer Design for Reaction Diffusion Equation With Large Output Delay

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ABSTRACT

The goal of this paper is to present a new finite-dimensional adaptive observer for some uncertain linear parabolic systems with delayed measurements. The observer is based on the modal decomposition approach and uses a classical persistent excitation condition. We prove that by using a finite-dimensional state predictor, we ensure exponential convergence of both states parameters estimation error for arbitrary long constant delay.

1 | Introduction

Reaction-diffusion equations model various physical phenomena, including temperature profile in the rod, concentration in chemical reactors, and air polluted areas [1] and [2]. On the other hand, throughout these last years, important attention has been paid to observer design for parabolic systems with uncertainties and/or with delayed measurements. To deal with this problem, two approaches exist in the literature. The first one is based on the PDE model. We can cite the recent work [3] and references therein, where a chain of observers in PDE form has been proposed for a class of parabolic systems to cope with a large constant delay and the works [4–7] for uncertain parabolic systems without delay. The second approach is based on the modal decomposition, where only a finite-dimensional part of the system is considered in the design of the observer. For this approach, we can cite the work [8] where an adaptive finite-dimensional observer was proposed for a class of distributed systems with some properties on the residual infinite-dimensional part. Recently the authors of [9] have considered the case of the heat equation, without uncertainties and with the sensor subjected to the time-varying delay case. An exponential convergence condition involving the bound of the delay has been derived with linear matrix inequalities (LMIs). The main advantage of this approach compared to PDE ones is in the fact that the proposed observer is based on a finite number of ODEs, which is more suitable for implementation than the PDE models. It has to be noticed that this approach was also used in [10], to derive an output feedback based on a finite-dimensional observer with time-varying output delay. Recently, in [11], the authors extend the finite-dimensional approach to parabolic uncertain systems where the uncertainties are located at output equations with fast time-varying delay. The exponential convergence of the adaptive finite-dimensional observer proposed in [11] was derived, provided that the bound of the delay is sufficiently small. In [12], the case where the uncertainties are also located in the state equation has been considered. An important challenge of [12], is that it treats the case when the function $\phi_1(\cdot, t)$ of the unknown part in the PDE model depends on the spatial variable and $\phi_1(\cdot, t) \in L^2(0,1)$ and also consider the fast time-varying delay at the output. On the other hand, in [13], the authors consider the case where the uncertainties are located in the state equation with rather restrictive conditions on the function $\phi_1(\cdot, t)$ and a constant delay. In the work [13], the function $\phi_1(\cdot, t)$ is assumed as a finite sum involving a finite number of eigenfunctions in $L^2(0,1)$, which is obviously a rather restrictive condition. To cope with constant delays, the authors use a classical output predictor. The exponential convergence is derived by using the classical small gain approach under restrictive conditions on the delay, which has to be sufficiently small. It has also to be noticed that the same restriction on the bound of the delay exists in [14] where the distributed delay case is considered. In both papers [13] and [14], the number of gains involved in the observers are at least equal to the number of terms in the finite sum. This property can imply a high number of gains on the observer even if the number of unstable modes is small. The goal of the present paper is to provide a more efficient solution than [13] by using a different approach. More precisely, we propose to use a finite-dimensional state predictor to handle large delay, and we provide a new exponentially convergent adaptive finite-dimensional observer. We also propose an algorithm where the number of gains involved in the observer is equal to the number of unstable modes of the systems.

Notations

We denote with $C(a,b)$ the linear space of continuous functions on the domain (a,b). $L^2(a,b)$ is the linear space of square integrable functions on (a,b). Finally, $H^1(a,b)$ is the Sobolev space of functions in $L^2(a,b)$ such that the function and its first derivative has a finite L^2 norm. Denote $A_n = \text{diag}(-\mu_1, \dots, -\mu_n)$, $C_n = (\sqrt{2}, \dots, \sqrt{2})$ is a row that has n columns, $L_n = (l_1, \dots, l_n)^T$ and, $p_n = (p_1, \dots, p_n)^T$ are vectors with n rows. $A_{i-j} = \text{diag}(-\mu_{i+1}, \dots, -\mu_j)$, $p_{i-j} = (p_{i+1}, \dots, p_j)^T$, $C_{n-j} = (\sqrt{2}, \dots, \sqrt{2})$ is a row that has $n-j$ columns.

2 | System Description and Assumptions

Consider the class of parabolic systems:

$$u_t = u_{xx} + qu + \phi_1(x,t)\theta \quad \text{for } t > 0,\ x \in (0,1) \tag{1a}$$

$$u_x(0) = u(1) = 0 \tag{1b}$$

$$u(x,0) = u_0 \tag{1c}$$

with initial condition $u_0 \in H^1(0,1)$ with $u_0(1) = 0$ and under the delayed measurement

$$y(t) = u(0, t-r) + \phi(t-r)\theta \tag{2}$$

where r is an arbitrary known constant delay. The constant q is a positive known parameter, ϕ and ϕ_1 are known and continuous functions that satisfy

$$|\phi(t)| \le M_\phi, \forall t \ge 0 \tag{3}$$

and

$$|\phi_1(x,t)| \le M_{\phi_1} \forall x \in [0,1], t \ge 0 \tag{4}$$

with some positive constants M_ϕ and M_{ϕ_1}.

We also suppose that $\phi_1(.,t) \in H^1(0,1)$ with $\phi_1(1,t) = 0$. The vector $\theta \in \mathbf{R}^m$ is a vector of unknown parameters. The term $\phi(t)\theta$ models either sensor uncertainties or faults to be detected and isolated. This uncertain term induces a difference between $u(0,t)$ and the available measurement $y(t)$. The term $\phi_1(x,t)\theta$ models uncertainties or disturbances that also disturb the model. The role of the adaptive observer is to provide an accurate estimation of both unmeasurable state $u(x,t)$ and the unknown vector of parameters θ.

The well-known regular Sturm-Liouville eigenvalue problem $\psi''(x) + \lambda\psi(x) = 0, \quad x \in [0,1]$ with $\psi(1) = \psi'(0) = 0$, generates an increasing sequence of eigenvalues $\lambda_n = \frac{\pi^2}{4}(2n-1)^2 \quad n \ge 1$ with corresponding eigenfunctions $\psi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x)$ for $n \ge 1$. The eigenfunctions ψ_n form an orthonormal basis of $L^2(0,1)$ with $f'(0) = f(1) = 0$. A strong solution of (1) is a function $u \in L^2(0,\infty); H^2(0,1) \cap C([0,\infty); H^1(0,1)$ and $u_t \in L^2((0,\infty); L^2(0,1))$ that satisfies (1c) for $t = 0$ and (1a), (1b) for almost all $t > 0$. By [15, Th. 7.7], (1) has a unique strong solution for $u_0 \in H^1(0,1)$ s.t $\quad u_0(1) = 0$. Consequently the solutions of the Equation (1) can be presented as

$$u(x,t) = \sum_{n=1}^{\infty} z_n(t)\psi_n(x) \tag{5}$$

where $z_n(t) = \int_0^1 u(x,t)\psi_n(x)dx$. We assume that the function ϕ_1 can be written as a finite sum as follows:

$$\phi_1(x,t) = \sum_{n=1}^{N_\phi} p_n(t)\psi_n(x) \tag{6}$$

where N_ϕ is a positive constant and $p_n(t) = \int_0^1 \phi_1(x,t)\psi_n(x)dx$. Notice that since both ϕ_1 and ψ_n are bounded, then p_n is also bounded and satisfies $|p_n| \le \sqrt{2}M_{\phi_1}$.

Differentiating the modes z_n and further integrating by parts twice, we have

$$\dot{z}_n(t) = -\lambda_n z_n(t) + qz_n(t) + p_n(t)\theta \quad n = 1, 2, \dots, N_\phi \tag{7}$$

and

$$\dot{z}_n(t) = -\lambda_n z_n(t) + qz_n(t) \quad n = N_\phi + 1, \dots, \infty \tag{8}$$

The output y can also be expressed as follows:

$$y(t) = \sqrt{2}\sum_{n=1}^{\infty} z_n(t-r) + \phi(t-r)\theta \tag{9}$$

Since λ_n is an increasing sequence, then we can define an integer N_0 as the smallest integer n for which the following inequality holds:

$$-\lambda_n + q < 0, \forall n > N_0 \tag{10}$$

We assume additionally that $q \ne \lambda_n$. Since (A_{N_0}, C_{N_0}) is observable [10], we choose L_{N_0} such that $A_{N_0} - L_{N_0}C_{N_0}$ is Hurwitz. Consider the state estimation errors $\tilde{Z}_N = (\tilde{z}_1, \dots, \tilde{z}_N)^T$, the delayed state estimation errors $\tilde{\xi}_{N_0} = (\tilde{\xi}_1, \dots, \tilde{\xi}_{N_0})^T$ and $\tilde{\xi}_{N-N_0} = (\tilde{\xi}_{N_0+1}, \dots, \tilde{\xi}_N)^T$ where $\tilde{\xi}_n(t) = \hat{x}_n(t) - z_n(t-r)$, and

$\tilde{Z}_{N-N_0} = (\tilde{z}_{N_0+1}, \ldots, \tilde{z}_N)^T$ where $\tilde{z}_n(t) = \hat{z}_n(t) - z_n(t)$, and the estimation parameter error $\tilde{\theta} = \hat{\theta} - \theta$.

Now let us consider the term

$$\zeta(t) = \sum_{n=N+1}^{\infty} z_n(t-r) \tag{11}$$

where $N = \max\{N_0, N_\phi\}$. It was already proven in [10], that $\sum_{n=N+1}^{\infty} \lambda_n z_n^2$ is well defined for any $t > t_0$ and $\sum_{n=N+1}^{\infty} \lambda_n z_n^2 \leq \|u_x(.,t)\|^2_{L^2(0,1)}$. We also have the following property:

$$\zeta(t) = \frac{1}{\sqrt{2}}\left(u(0,t-r) - \sum_{=1}^{N} z_n(t-r)\psi_n(0)\right)$$

then from the fact that $u(0,t-r) = -\int_0^1 u_x(x,t-r)dx$ and $\psi_n(0) = -\int_0^1 \psi_n'(x)dx$, we deduce that

$$\zeta(t) = -\frac{1}{\sqrt{2}}\int_0^1 \left(u_x(x,t-r) - \sum_{=1}^{N} z_n(t-r)\psi_n'(x)\right)dx$$

since $u_x(x,t-r) = \sum_{n=1}^{\infty} z_n(t-r)\psi_n'(x)$, then

$$\zeta(t) = -\frac{1}{\sqrt{2}}\int_0^1 \left(\sum_{n=N+1}^{\infty} z_n(t-r)\psi_n'(x)\right)dx$$

using the Schwarz inequality, and since $\psi_n'(x) = -\sqrt{2\lambda_n}\sin(\sqrt{\lambda_n}x)$, then we deduce that

$$|\zeta(t)|^2 \leq \frac{1}{2}\sum_{n=N+1}^{\infty} z_n^2(t-r)\left\|\psi_n'(x)\right\|^2_{L^2(0,1)} \leq \frac{1}{2}\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t-r) \tag{12}$$

Consider the Lyapunov function

$$V(t) = \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t-r) \tag{13}$$

then its time derivative will be written as follows:

$$\dot{V}(t) = -2\sum_{n=N+1}^{\infty} \lambda_n \mu_n z_n^2(t-r) \tag{14}$$

we easily derive that

$$\dot{V}(t) \leq -2\mu_{N+1}\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t-r) = -2\mu_{N+1}V(t) \tag{15}$$

From this last inequality, we deduce that both $\sum_{n=N+1}^{\infty} \lambda_n z_n^2$ and $|\zeta(t)|$ converge exponentially to zero.

3 | Finite-Dimensional Adaptive Observer Design

3.1 | Adaptive Observer Structure

Following [10], we will construct an N-dimensional adaptive observer constituted by two parts. The first one is an adaptive observer which provides an estimation of the delayed state $u(x,t-r)$ and has the following structure

$$\begin{cases} \dot{\xi}_n(t) = -\mu_n\xi_n(t) + p_n(t-r)\hat{\theta} - l_n(\hat{y} - y) + v_1 & n = 1, \ldots, N_0 \\ \dot{\xi}_n(t) = -\mu_n\xi_n(t) + p_n(t-r)\hat{\theta} + v_2 & n = N_0+1, \ldots, N \\ \hat{y}(t) = \sqrt{2}\sum_{n=1}^{N}\xi_n(t) + \phi(t-r)\hat{\theta}(t) \end{cases} \tag{16}$$

with N_0 defined in (10) and $N = \max\{N_0, N_\phi\}$, $\mu_n = \lambda_n - q$, l_n are observer gains, $\hat{\theta}(t)$ is estimate of θ, v_1 and v_2 are additional signals that we will choose later on.

The second part is a finite-dimensional state predictor which recovers the state $u(x,t)$ and is constituted by two sub-parts: The first one is a sequence of predictors $\hat{Z}^{(i)}_{N_0}(t) \in \mathbb{R}^{N_0}$ with $\hat{Z}^{(i)}_{N_0}(t) = \mathrm{col}_{n=1}^{N_0}(\hat{z}_n(t-i\delta))$ which estimates the vector of the delayed states $Z_{N_0}(t-i\delta) = \mathrm{col}_{n=1}^{N_0}(z_n(t-i\delta))$ of the unstable part and defined for $i = 0, \ldots v_r - 1$ by

$$\begin{aligned} \dot{\hat{Z}}^{(i)}_{N_0}(t) &= A_{N_0}\hat{Z}^{(i)}_{N_0}(t) + \mathrm{col}_{n=1}^{N_0}(p_n(t-i\delta))\hat{\theta}(t) \\ &\quad - e^{\bar{A}_{\alpha_i,N_0}\delta}L_{N_0}C_{N_0}(\hat{Z}^{(i+1)}_{N_0}(t) - \hat{Z}^{(i)}_{N_0}(t-\delta)) \\ \hat{Z}^{(v_r)}_{N_0}(t) &= \xi_{N_0}(t) \end{aligned} \tag{17}$$

with $\xi_{N_0}(t) = \mathrm{col}_{n=1}^{N_0}(\xi_n(t))$, L_{N_0} such that $\bar{A}_{\alpha_i,N_0} = \bar{A}_{N_0} + \alpha_i I_{N_0}$ is Hurwitz, $\bar{A}_{N_0} = A_{N_0} - L_{N_0}C_{N_0}$, and v_r an integer such that $\delta = r/v_r$ satisfies

$$\int_0^\delta \left\|e^{(\bar{A}_{N_0}+\alpha I_{N_0})s}L_{N_0}C_{N_0}\right\| \mathrm{d}s < 1 \tag{18}$$

The second sub-part is represented as follows:

$$\hat{z}_n(t) = e^{-\mu_n r}\xi_n(t) + \int_{t-r}^{t} e^{-\mu_n(t-s)}p_n(s)\hat{\theta}(s)ds, \ n = N_0+1, \ldots, N \tag{19}$$

and estimates the states $z_n(t)$ for $i = N_0+1, \ldots, N$ which have a finite number of stable modes. The overall finite-dimensional state predictor has the following structure:

$$\hat{u}(x,t) = \sum_{n=1}^{N}\hat{z}_n(t)\psi_n(x) \tag{20}$$

Remark 1. We can easily remark that we use two kinds of predictors: one for unstable modes and another one for stable modes. This is due to the fact that for unstable modes, the predictor in integral form exhibits instability in the implementation, which is not the case for stable modes.

Lemma 1. *If $\zeta(t)$, $\tilde{\xi}_{N_0}(t)$ and $\tilde{\theta}(t)$ are exponentially stable with rate ρ, and suppose that the following inequality is fulfilled*

$$\int_0^\delta \left\|e^{(\bar{A}_{N_0}+\alpha I_{N_0})s}L_{N_0}C_{N_0}\right\| \mathrm{d}s < 1 \tag{21}$$

Then for an arbitrary positive constant $\alpha < \rho$, the sequence of predictors defined in (17) for an arbitrary sequence $\alpha_0 < \alpha_1 < \ldots < \alpha$ is such that $|\mathrm{col}_{n=1}^{N_0}(z_n(t-i\delta)) - \hat{Z}^{(i)}_{N_0}(t)|$ is exponentially stable *with rate α_i. In particular, $|\hat{Z}^{(0)}_{N_0}(t) - \text{col}_{n=1}^{N_0}(z_n(t))| \to 0$ exponentially with rate α_0.*

Proof. Denoting $\varepsilon^{(i)}_{N_0}(t) = Z_{N_0}(t - i\delta) - \hat{Z}^{(i)}_{N_0}(t)$ the estimation error of the i-th predictor, we have

$$\dot{\varepsilon}^{(i)}_{N_0}(t) = A_{N_0}\varepsilon^{(i)}_{N_0}(t) - \text{col}_{n=1}^{N_0}(p_n(t - i\delta))\tilde{\theta}(t) + e^{\bar{A}_{\alpha_i,N_0}\delta}L_{N_0}C_{N_0}(\varepsilon^{(i+1)}_{N_0}(t) - \varepsilon^{(i)}_{N_0}(t - \delta)) \tag{22}$$

Define the scaled version of $\varepsilon^{(i)}_{\alpha_i,N_0}(t) = e^{\alpha_i t}\varepsilon^{(i)}_{N_0}(t)$. Clearly, boundedness of $\varepsilon^{(i)}_{\alpha_i,N_0}(t)$ implies that $\varepsilon^{(i)}_{N_0}(t)$ is exponentially stable with rate α_i.

$$\dot{\varepsilon}^{(i)}_{\alpha_i,N_0}(t) = (A_{N_0} + \alpha_i I_{N_0})\varepsilon^{(i)}_{\alpha_i,N_0}(t) - e^{\alpha_i t}\text{col}_{n=1}^{N_0}(p_n(t - i\delta))\tilde{\theta}(t) - e^{\bar{A}_{\alpha_i,N_0}\delta}L_{N_0}C_{N_0}(e^{\alpha_i t}\varepsilon^{(i+1)}_{\alpha_{i+1},N_0}(t) - \varepsilon^{(i)}_{\alpha_i,N_0}(t - \delta)) \tag{23}$$

Since $\alpha_i < \alpha_{i+1} < \alpha < \rho$, in the inductive hypothesis that $\varepsilon^{(i+1)}_{N_0}(t)$ is exponentially stable with rate α_{i+1}, which is also the case for $\tilde{\xi}_{N_0}$ with rate ρ, this equation becomes

$$\dot{\varepsilon}^{(i)}_{\alpha_i,N_0}(t) = (A_{N_0} + \alpha_i I_{N_0})\varepsilon^{(i)}_{\alpha_i,N_0}(t) - e^{\bar{A}_{\alpha_i,N_0}\delta}L_{N_0}C_{N_0}(\varepsilon^{(i)}_{\alpha_i,N_0}(t - \delta)) \tag{24}$$

that admits the integral representation

$$\varepsilon^{(i)}_{\alpha_i,N_0}(t) = \int_{t-\delta}^{t} e^{\bar{A}_{\alpha_i,N_0}(t-s)}L_{N_0}C_{N_0}\varepsilon^{(i)}_{\alpha_i,N_0}(s)\,\mathrm{d}s + \kappa_i \tag{25}$$

where κ_i depends on the initial conditions, as it can be verified by differentiation. Thus,

$$|\varepsilon^{(i)}_{\alpha_i,N_0}(t)| \le \int_0^{\delta} e^{(\alpha_i-\alpha)s}\left|e^{\bar{A}_{\alpha,N_0}\tau}L_{N_0}C_{N_0}\right|\mathrm{d}\tau \cdot \sup_{\tau\in[t-r,t]}|\varepsilon^{(i)}_{\alpha_i,N_0}(\tau)| + |\kappa_i| \tag{26}$$

Since $e^{(\alpha_i-\alpha)s} < 1$, the integral term is less than 1, which guarantees the boundedness of $|\varepsilon^{(i)}_{\alpha_i,N_0}(t)|$ and the exponential stability with rate α_i of $|\varepsilon^{(i)}_{N_0}(t)|$. □

Remark 2. The observer proposed in [13] which supposes $\phi(t) = 0$, has the following structure:

$$\begin{cases} \dot{\hat{z}}_n(t) = -\mu_n\hat{z}_n(t) + p_n(t)\hat{\theta}(t) - l_n(\hat{y} - y_p(t + r)) + v_1 \\ \quad n = 1, \ldots, N_\phi \\ \hat{y}(t) = \sqrt{2}\sum_{n=1}^{N}\hat{z}_n(t) \end{cases} \tag{27}$$

with y_p in Equation (22) of [13] is classical output predictor by using the classical formula $y(t+r) = y(t) + \int_0^r \dot{y}(t+s)ds$. As we can see, the above structure is based on the prediction of the delayed output by using a classical predictor (Equation (22) of [13]) based on the Leibniz formula. This predictor ensures convergence only for small delays and leads to a conservative condition on the bound of delay r. In the present paper we adopt a different approach where we have two parts: The first one is constituted by an adaptive observer that provides an estimation of the delayed states, and a second one, which is based on a sequence of predictors that can recover the value of state $u(x,t)$ at instant t for any arbitrarily long delay r.

Remark 3. It has to be noticed that the states predictors (19) can also be written as delayed differential equations (dde):

$$\dot{\hat{z}}_n(t) = e^{-\mu_n r}\dot{\xi}_n(t) - \mu_n[\hat{z}_n(t) - e^{-\mu_n r}\xi_n(t)] + p_n(t)\hat{\theta}(t) - e^{-\mu_n r}p_n(t - r)\hat{\theta}(t - r) \tag{28}$$

with initial condition

$$z_n(0) = e^{-\mu_n r}\xi_n(0) + \int_{-r}^{0} e^{\mu_n s}p_n(s)\hat{\theta}(s)ds$$

The estimation errors system $\tilde{\xi}_{N_0}(t)$ and $\tilde{\xi}_{N-N_0}(t)$ can be expressed as follows:

$$\begin{aligned} \dot{\tilde{\xi}}_{N_0}(t) &= (A_{N_0} - L_{N_0}C_{N_0})\tilde{\xi}_{N_0}(t) \\ &\quad - (L_{N_0}\phi(t - r) - p_{N_0}(t - r))\tilde{\theta}(t) \\ &\quad - Ł_{N_0}C_{N-N_0}\tilde{\xi}_{N-N_0}(t) + L_{N_0}\sqrt{2}\zeta(t) + v_1 \\ \dot{\tilde{\xi}}_{N-N_0}(t) &= A_{N-N_0}\tilde{\xi}_{N-N_0}(t) + p_{N-N_0}(t - r)\tilde{\theta}(t) + v_2 \end{aligned} \tag{29}$$

Consider for $\tilde{\xi}_{N_0}$ and $\tilde{\xi}_{N-N_0}$ defined in (29) the decoupling transformations [16]

$$e_{N_0}(t) = \tilde{\xi}_{N_0}(t) - \alpha_1(t)\tilde{\theta}(t) \tag{30}$$

and

$$e_{N-N_0}(t) = \tilde{\xi}_{N-N_0}(t) - \alpha_2(t)\tilde{\theta}(t) \tag{31}$$

where α_1 is the solution of an auxiliary filter which is defined as follows:

$$\begin{cases} \dot{\alpha}_1(t) = (A_{N_0} - L_{N_0}C_{N_0})\alpha_1(t) - (L_{N_0}\phi(t - r) - p_{N_0}(t - r)) \\ \qquad + L_{N_0}C_{N-N_0}\alpha_2(t) \\ v_1(t) = \alpha_1(t)\dot{\hat{\theta}} \end{cases} \tag{32}$$

and α_2 is the solution of an auxiliary filter which is also defined as follows:

$$\begin{cases} \dot{\alpha}_2(t) = A_{N-N_0}\alpha_2(t) + p_{N-N_0}(t - r) \\ v_2(t) = \alpha_2(t)\dot{\hat{\theta}} \end{cases} \tag{33}$$

From this, we deduce two ODEs for e_{N_0} and e_{N-N_0} which do not depend on $\tilde{\theta}$.

$$\dot{e}_{N_0}(t) = (A_{N_0} - L_{N_0}C_{N_0})e_{N_0}(t) + \sqrt{2}\,L_{N_0}\zeta(t) - L_{N_0}C_{N-N_0}e_{N-N_0}(t) \tag{34}$$

and

$$\dot{e}_{N-N_0}(t) = (A_{N-N_0})e_{N-N_0}(t) \tag{35}$$

Since $N > N_0$, then A_{N-N_0} is Hurwitz and consequently e_{N-N_0} is exponentially stable. Consider

$$V_0(t) = e_{N_0}(t)^T P_0 e_{N_0}(t) \tag{36}$$

where P_0 satisfies the following inequality

$$P_0(A_{N_0} - L_{N_0}C_{N_0}) + (A_{N_0} - L_{N_0}C_{N_0})^T P_0 \leq -2P_0 \tag{37}$$

It's time-derivative will be expressed as follows:

$$\begin{aligned}\dot{V}_0 &\leq -2V_0 \\ &+ 2e_{N_0}(t)^T P_0\left(\sqrt{2}\, L_{N_0}\zeta(t) - L_{N_0}C_{N-N_0}e_{N-N_0}(t)\right)\end{aligned} \tag{38}$$

Applying again Young's inequality, then

$$\begin{aligned}\dot{V}_0 &\leq -2V_0 + \epsilon_1|e_{N_0}(t)|^2 + \frac{4}{\epsilon_1}|P_0|^2||L_{N_0}|^2|\zeta(t)|^2 \\ &+ \frac{2|P_0|^2}{\epsilon_1}|L_{N_0}C_{N-N_0}|^2|e_{N-N_0}(t)|^2\end{aligned} \tag{39}$$

which gives us

$$\dot{V}_0 \leq -\left(2 - \frac{\epsilon_1}{\lambda_{\min}(P_0)}\right)V_0 + \frac{4}{\epsilon_1}|P_0|^2||L_{N_0}|^2|\zeta(t)|^2 \tag{40}$$

$$+ \frac{2|P_0|^2}{\epsilon_1}|L_{N_0}C_{N-N_0}|^2|e_{N-N_0}(t)|^2 \tag{41}$$

Choosing $\epsilon_1 = \lambda_{\min}(P_0)$, then

$$\dot{V}_0 \leq -V_0 + \frac{4}{\lambda_{\min}(P_0)}|P_0|^2||L_{N_0}|^2|\zeta(t)|^2 \tag{42}$$

$$+ \frac{2|P_0|^2}{\lambda_{\min}(P_0)}|L_{N_0}C_{N-N_0}|^2|e_{N-N_0}(t)|^2 \tag{43}$$

Since both $e_{N-N_0}(t)$ and $|\zeta(t)|$ are exponentially vanishing, we can also deduce from the comparison lemma that $|e_{N_0}|$ is also exponentially vanishing.

3.1.1 | Estimation Law Design

Following [16], we propose the following estimation law:

$$\dot{\hat{\theta}}(t) = -R(t)(\alpha^T(t)C_N^T + \phi^T(t-r))(\hat{y}(t) - y(t)) \tag{44}$$

with

$$\begin{aligned}\frac{\mathrm{d}R(t)}{dt} &= R(t) - R(t) \\ &\times (\alpha^T(t)C_N^T + \phi^T(t-r))(C_N\alpha^T(t) + \phi(t-r))R(t)\end{aligned} \tag{45}$$

where $\alpha^T = (\alpha_1^T\ \alpha_2^T)$ and $C_N = (C_{N_0}\ C_{N-N_0})$. It was already proven in [16] that if $|\alpha|$ and $|\phi|$ are bounded and if the persistent excitation condition

$$\int_t^{t+T} K^T(s)K(s)ds \geq \beta_0\mathbb{I} \tag{46}$$

with

$$K^T(t) = (\alpha^T(t)C_N^T + \phi^T(t-r)) \tag{47}$$

holds for some positive constant β_0, then both $R(t)$ and $R^{-1}(t)$ are positive definite matrices and there exist two positive constants β_1 and β_2 such that the following inequalities hold:

$$\beta_1\mathbb{I}_m \leq R(t) \leq \beta_2\mathbb{I}_m \tag{48}$$

and the inverse matrix satisfies:

$$\begin{aligned}\frac{\mathrm{d}R^{-1}(t)}{dt} &= -R^{-1}(t) \\ &+ (\alpha^T(t)C_N^T + \phi^T(t-r))(C_N\alpha^T(t) + \phi(t-r))\end{aligned} \tag{49}$$

with

$$\beta_1\mathbb{I}_m \leq R^{-1}(t) \leq \beta_2\mathbb{I}_m \tag{50}$$

3.1.2 | Convergence Analysis

The parameter estimation error is governed by the following ODEs:

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -R(t)K^T(t)K(t)\tilde{\theta}(t) \\ &- R(t)K^T(t)(C_{N_0}\epsilon_{N_0}(t) + C_{N-N_0}e_{N-N_0}(t) - \sqrt{2}\zeta(t))\end{aligned} \tag{51}$$

To study the convergence of $\tilde{\theta}$, let us consider the following Lyapunov function for (51):

$$V_\theta(t) = \tilde{\theta}^T(t)R^{-1}(t)\tilde{\theta}(t) \tag{52}$$

Then the time-derivative of V_θ satisfies the following equality

$$\dot{V}_\theta(t) = -V_\theta(t) + |C_{N_0}\epsilon_{N_0}(t) + C_{N-N_0}e_{N-N_0}(t) - \sqrt{2}\zeta(t)|^2 \tag{53}$$

Using Young's inequality, we derive

$$\begin{aligned}\dot{V}_\theta(t) &\leq -V_\theta(t) + 4|C_{N_0}\epsilon_{N_0}(t)|^2 + 4|C_{N-N_0}e_{N-N_0}(t)|^2 \\ &+ 8|\zeta(t)|^2\end{aligned} \tag{54}$$

Since e_{N-N_0}, e_{N_0} and ζ converge also exponentially to zero, then by applying the comparison lemma to (54), we conclude that $\tilde{\theta}$ will also converge exponentially to zero. On the other hand, from (30) and (31), we can deduce that $|\tilde{\xi}_N(t)|^2 \leq 2|\epsilon_N(t)|^2 + 2|\alpha(t)|^2|\tilde{\theta}(t)|^2$. Since both systems (32) and (33) are ISS and $|p_N|$ and $|\phi|$ are bounded, then $|\alpha_1|$ and $|\alpha_2|$ are also bounded. This allows us to conclude that $|\tilde{\xi}_N|$ is also exponentially convergent to zero. Now let us consider the prediction error

$$\tilde{z}_n(t) = e^{-\mu_n r}\tilde{\xi}_n(t) + \int_{t-r}^{t} e^{-\mu_n(t-s)}p_n(s)\tilde{\theta}(s)ds \tag{55}$$

Since both $\tilde{\xi}_n$ and $\tilde{\theta}(t)$ converge exponentially to zero and p_n are bounded terms, then we can say that there exist two positive constants M_0 and σ_0 such that $|\tilde{\theta}(t)| + |\tilde{\xi}_n| \leq M_0 e^{-\sigma_0 t}$, $n = 1, \ldots, N$. From this, we can deduce that we can derive the following inequality for $n = N_0 + 1, \ldots, N$

$$|\tilde{z}_n(t)| \leq e^{-\mu_n r}|\tilde{\xi}_n(t)| + M_\phi M_0 e^{-\mu_n t}\int_{t-r}^{t} e^{-(\sigma_0-\mu_n)s}ds \tag{56}$$

which gives us

$$|\tilde{z}_n(t)| \leq e^{-\mu_n r}|\tilde{\xi}_n(t)| + M_\phi M_0 e^{-\mu_n t}\int_{t-r}^{t} e^{-(\sigma_0-\mu_n)s}ds \tag{57}$$

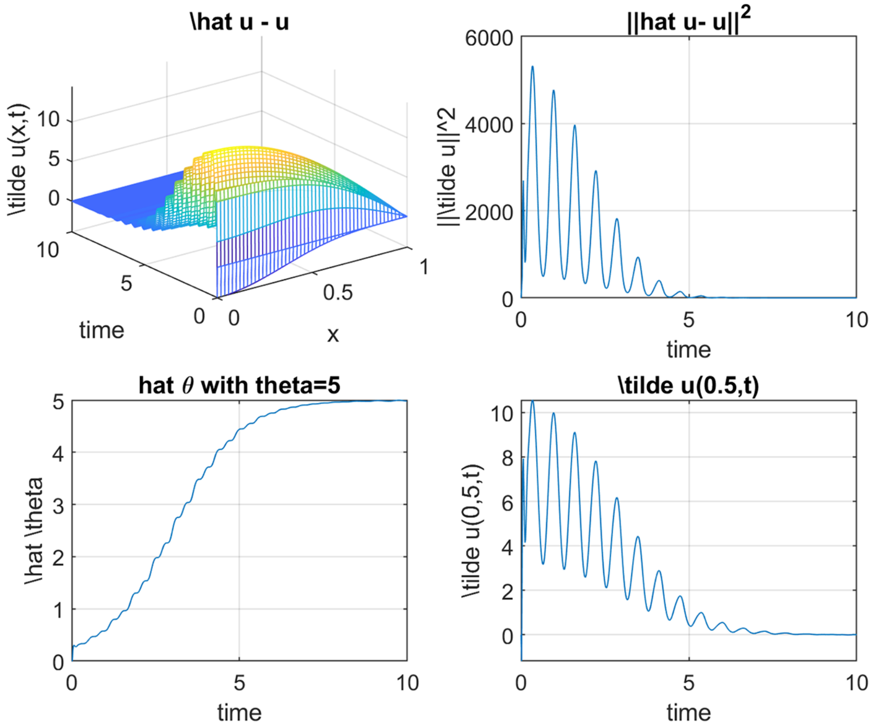

FIGURE 1 | Estimation errors for system (1) with delay $r = 6$ and $\theta = 5$.

and

$$|\tilde{z}_n(t)| \leq e^{-\mu_n r} M_0 e^{-\sigma_0 t} + \frac{M_\phi M_0 (e^{(\sigma_0 - \mu_n) r} - 1)}{\sigma_0 - \mu_n} e^{-\sigma_0 t} \tag{58}$$

which means that

$$|\tilde{Z}_{N-N_0}(t)|^2 \leq 2e^{-2\sigma_0 t} \sum_{n=N_0+1}^{N} \left(e^{-2\mu_n r} M_0^2 + \frac{M_\phi^2 M_0^2 (e^{(\sigma_0 - \mu_n) r} - 1)^2}{(\sigma_0 - \mu_n)^2} \right)$$

On the hand, since $\zeta(t)$, $\tilde{\theta}(t)$, and $\xi_{N_0}(t)$ are exponentially vanishing, we deduce from Lemma (1) that there exist positive constants α_0, and M_0 such that $|\hat{Z}^{(0)}_{N_0}(t) - \mathrm{col}_{n=1}^{N_0}(z_n(t))| = |\tilde{Z}_{N_0}|^2 \leq M_0 e^{-\alpha_0 t}$. and from Parseval's equality, we can say that

$$\|\hat{u}(\cdot,t) - u(\cdot,t)\|^2_{H^1(0,1)} = \sum_{i=1}^{N} \lambda_i \tilde{z}_i^2(t) + \sum_{n \geq N+1} \lambda_n z_n^2(t) \tag{59}$$

Since $\sum_{i=1}^{N} \lambda_i \tilde{z}_i^2(t) \leq \lambda_N |\tilde{Z}_N|^2$ and since $\sum_{n\geq N+1} \lambda_n z_n^2(t)$ is exponentially vanishing, then we can also conclude that the H^1 norm $\|\hat{u}(\cdot,t) - u(\cdot,t)\|_{H^1(0,1)}$ converges exponentially to zero.

Theorem 1. *Consider system (1) with initial condition $u_0 \in H^1(0,1), u_0(1) = 0$, and adaptive observer described by (16), (17), (19), (20), (32), (33), and (44). Let $N_0 \in \mathbb{N}$ satisfy (10) and $N = \max\{N_0, N_\phi\}$. Let the vector of gains $L_{N_0} = (l_1, \ldots, l_{N_0})^T$ satisfy (37). Let the integer v_r and the positive constant α satisfy (21). Then under persistent excitation conditions (46), the norms $||\tilde{u}(.,t)||_{H^1(0,1)}$ and $|\tilde{\theta}(t)|$ converge exponentially to zero.*

Remark 4. Compared to the result of [13], we can easily remark two important features: (i) Contrarily to the observer proposed in [13], we cope with an arbitrary long constant delay, provided that the integer v_r is chosen sufficiently large, and ensure exponential convergence of both state and parameter estimation errors. (ii) Another important advantage of our observer compared to [13] and [14] is in the fact that the number of the observer's gains in our observer is fixed by the number of unstable modes N_0, contrarily to the observer of [13] and [14], which runs with N_ϕ gains. If N_ϕ is high, this implies a high number of observer's gains in [13] and [14], even if the number of unstable modes is not important.

4 | Example

In this section we illustrate our result on system (1). We consider $q = 3$ and $\phi_1(x,t) = \cos(\pi\ x/2) + \sin(t)\cos(3\pi\ x/2)$. The output $y = u(0,t) + (2 - \cos(10t))\theta$. The condition (10) gives us $N_0 > \frac{1}{2} + \frac{\sqrt{3}}{\pi}$, then the smallest integer satisfying (10) is $N_0 = 2$. Given $N_\phi = 2$, we deduce that the smallest integer is $N = 2$. The simulations are performed with $N_0 = 2$ and $N = 2$, $v_r = 30$, $L = (23.2, \quad 1.1)^T$ and $r = 6$ (Figure 1) .

5 | Conclusion

In this paper, we presented a new finite-dimensional adaptive observer for a class of linear parabolic systems with a free delay at the output. Our algorithm ensures exponential convergence to zero of both state and parameter estimation errors for a free constant delay.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

References

1. P. Christofides, *Nonlinear and Robust Control Of PDE Systems: Methods and Applications to Transport Reaction Processes*, vol. 55 (Berlin, Germany: Springer, 2001), B29–B30.

2. R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory* (New York: Springer, 1995).

3. T. Ahmed-Ali, E. Fridman, F. Giri, M. Kahelras, F. Lamnabhi-Lagarrigue, and L. Burlion, "Observer Design for a Class of Parabolic Systems With Large Delays and Sampled Measurements," *IEEE Transactions on Automatic Control* 5-65 (2020): 2200–2206.

4. T. Ahmed-Ali, F. Giri, M. Krstic, L. Burlion, and F. Lamnabhi-Lagarrigue, "Adaptive Boundary Observer for Parabolic Pdes Subject to Domain and Boundary Parameter Uncertainties," *Automatica* 72 (2016): 115–122.

5. P. Ascencio, "Adaptive Observer Design for Parabolic Partial Differential Equations," 2017. *PHD Thesis*.

6. M. A. Demetriou, J. Baumeister, W. Scondo, and I. G. Rosen, "Asymptotic Embedding Methods for Parameter Estimation," *in Proceedings of the 26th IEEE Conference on Decision and Control*(1987): 170–174.

7. A. Smyshlyaev and M. Krstic, "Output-Feedback Adaptive Control for Parabolic PDES With Spatially Varying Coefficients," in *Proceedings. 45th IEEE Conference on Decision and Control* (2006): 3099–3104.

8. J. H. Lilly, "Finite-Dimensional Adaptive Observers Applied to Distributed Parameter Systems," *IEEE Transactions on Automatic Control* 38 (1993): 469–474.

9. A. Selivanov and E. Fridman, "Boundary Observers for a Reaction-Diffusion System Under Time-Delayed and Sampled-Data Measurements," *IEEE Transactions on Automatic Control* 8 (2018): 3385–3390.

10. R. Katz and E. Fridman, "Constructive Method for Finite-Dimensional Observer-Based Control of 1-D Parabolic Pdes," *Automatica* 122 (2020): 109285.

11. T. Ahmed-Ali, E. Fridman, and F. Lamnabhi-Lagarrigue, "Finite-Dimensional Adaptive Observers Design for Reaction-Diffusion Systems," in *Proceedings of IEEE Conference on Decision and Control* (*CDC*) (2023): 8131–8136.

12. T. Ahmed-Ali, E. Fridman, and F. Lamnabhi-Lagarrigue, "Finite-Dimensional Adaptive Observer Design for Linear Parabolic Systems," *in Proceedings of 4 th IFAC ACNDC, Londres* (2024).

13. H. Rafia, A. Benabdelhadi, F. Giri, H. Ouadi, and F. Z. Chaoui, "Finite-Dimensional Output-Predictor-Based Adaptive Observer for Heat PDES With Sensor Delay," *International Journal of Adaptive Control and Signal Processing* 38-4 (2024): 1153–1171.

14. O. Ammari, F. Giri, M. Krstic, et al., "Adaptive Observer Design for Heat PDES With Sensor Delay and Parameter Uncertainties," *IEEE Transactions on Automatic Control*, Early Access (2024), https://doi.org/10.1109/TAC.2024.3422891.

15. J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDES and the Theory of Global Attractors* (London, UK: Cambridge University Press, 2001).

16. Q. Zhang, "Adaptive Observer for Multiple-Input- Multiple-Output (Mimo) Linear Time-Varying Systems," *IEEE Transactions on Automatic Control* 4, no. 43 (2002): 525–529.