Finite-Dimensional Observer-Based Boundary Control of 1-D Linear Parabolic-Elliptic Systems

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***Abstract*—This letter investigates the finite-dimensional observer-based boundary control for 1D linear parabolic-elliptic systems via the modal decomposition method. To address the potential multiple eigenvalues arising from the elliptic equation, we implement bilateral actuations (one Dirichlet and one Neumann) on the boundary of the parabolic equation with two point measurements. When the eigenvalues are simple, one boundary actuation and one point measurement are sufficient, but the second input and output may reduce the observer dimension. We present efficient LMI conditions for finding observer dimension, as well as controller and observer gains, ensuring the H^1 exponential stability with any desirable decay rate. We show that the LMIs are always feasible for large enough values of the observer dimension. Numerical examples demonstrate the efficiency of the method.**

***Index Terms*—Parabolic-elliptic system, observer-based control, modal decomposition, Lyapunov method.**

I. Introduction

PARABOLIC-ELLIPTIC systems appear in numerous applications, including lithium-ion cells [1], transport networks [2], chemotaxis phenomena [3], and thermistor [4]. Recently, the controllability and stabilization of parabolic-elliptic systems have been widely studied. In [3], [5] and [6], non-local and boundary null controllability of parabolic-elliptic systems were analyzed. In [7, Ch. 10], the stabilization of parabolic-elliptic systems was suggested in the context of boundary control of Kuramoto-Sivashinsky and Korteweg de Vries equations.

For linear parabolic-elliptic systems, boundary state-feedback control and output-feedback control based on PDE observer of were studied by the backstepping approach in [8], [9], [10]. In [11], finite-dimensional state-feedback boundary control with constant delay was investigated by the modal decomposition method under the assumption of simple eigenvalues. In [12], boundary stabilization of a class of singularly perturbed reaction-diffusion systems was achieved by stabilizing the corresponding reduced-order parabolic-elliptic system.

Finite-dimensional observer-based controllers for parabolic PDEs are attractive in practical applications and have been extensively studied since the 1980s [13], [14]. In the recent paper [15], a finite-dimensional observer-based control was suggested for 1D parabolic PDEs, where constructive LMI-based conditions were provided to determine the observer dimension. The complexity of these LMIs was subsequently reduced in [16]. Based on [15], some extensions have been done for both unbounded observer/controller operators [17], input/output delay robustness and delay compensation [16], ODE-heat equation cascade [18], and high-dimensional parabolic equations [19], [20], [21]. However, the finite-dimensional observer-based control for parabolic-elliptic systems remains open. The main challenges, compared to parabolic equations, arise from the elliptic equation, which can result in multiple eigenvalues, leading to an uncontrollable and unobservable finite-dimensional subsystem.

In this letter, for the first time, we study the finite-dimensional observer-based control of parabolic-elliptic systems via the modal decomposition method. To address the multiple eigenvalues caused by the elliptic equation, we consider bilateral actuations, one Dirichlet and one Neumann, on the boundary of the parabolic equation and two point measurements. We employ a dynamic extension with the corresponding proportional-integral controller and prove H^1 exponential stability. We prove controllability and observability of the finite-dimensional system, which is not straightforward for multiple eigenvalues case. Given a desirable decay rate, we provide LMI conditions for finding observer dimension N, controller and observer gains. We show that the LMIs are always feasible without restrictions on system parameters and decay rate, and if the LMIs are feasible for some N, they remain feasible for $N+1$. Moreover, if the eigenvalues are simple (as in [11]), our method is applicable for one actuation and one measurement. Numerical examples demonstrate the efficiency of our method. The contribution of this letter compared to the existing results is formulated as follows:

1) For the first time, we present finite-dimensional observer-based control for parabolic-elliptic systems, which is easier for implementation than the PDE observer in [8], [9].

2) Compared to [8], [9] based on the backstepping, for single input and single output, we have milder restrictions on system parameters and can achieve any desired decay rate.

3) Compared to state-feedback control via modal decomposition presented in [11] restricted to simple eigenvalues, we introduce additional and appropriate input and output to

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manage with the controllability and observability issues caused by multiple eigenvalues. Bilateral control is applicable not only in the multiple eigenvalues case, but has advantages over the single input (in the simple eigenvalue case) in terms of the observer dimension (see the end of the example) and in the case of the loss of one of the actuators.

Notations: Denote by $L^2(0,1)$ the space of all square integrable functions $f:[0,1]\to\mathbb{R}$ with inner product $\langle f,g\rangle=\int_0^1 f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|_{L^2}^2=\langle f,f\rangle$. $H^1(0,1)$ is the Sobolev space of functions $f:[0,1]\longrightarrow\mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(0,1)$ is $\|f\|_{H^1}^2=\|f\|_{L^2}^2+\|f'\|_{L^2}^2$. The Euclidean norm is denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is positive definite with symmetric elements denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^{\mathrm{T}}Px$. Denote by $\mathbb{N}$ the set of positive integers.

II. Main Results

A. System Under Consideration

We consider the following parabolic-elliptic system:

$$\begin{aligned} &z_t(x,t)=z_{xx}(x,t)-\rho z(x,t)+\alpha v(x,t),\\ &0=v_{xx}(x,t)-\gamma v(x,t)+\beta z(x,t),\\ &z(x,0)=z_0(x),\ \ x\in(0,1), \end{aligned}\tag{1}$$

with the boundary conditions

$$\begin{aligned} &z(0,t)=d_1\mathbf{u}_1(t),\, z_x(1,t)=d_2\mathbf{u}_2(t),\\ &v(0,t)=v_x(1,t)=0, \end{aligned}\tag{2}$$

where ρ,γ,α,β are all real with $\alpha\beta\neq0$, $\gamma\neq-\lambda_n$, $\lambda_n=(n-\frac{1}{2})^2\pi^2$, $n\in\mathbb{N}$, $\mathbf{u}_1(t),\mathbf{u}_2(t)\in\mathbb{R}$ are control inputs, $d_1,d_2\in\{0,1\}$. Let $d=d_1+d_2\geq1$, meaning that at least one actuation is active. Let z_0 be the initial state. Note that for any z_0, the initial condition $v(\cdot,0)$ is determined by solving the second equation in (1). Therefore, it is unnecessary to specify an initial condition for v.

Remark 1: The parabolic-elliptic system (1) can be interpreted as a simplified model for lithium-ion batteries, where z is the concentration of lithium ions and v is the electric potential in the electrolyte [1]. The parabolic equation models ion diffusion with terms for ion depletion ρz and coupling to the electric potential αv. The steady-state elliptic equation characterizes the electric potential response to ion concentration. The Dirichlet actuation typically represents a controlled concentration or flux of lithium ions entering the system, while the Neumann actuation governs the flux or gradient of ion concentration. The stabilization of (1) has been studied in [8], [9] under Neumann and in [10], [11], [12] under Dirichlet actuations.

We consider one or two sensors at points $x_1^*,x_2^*\in(0,1]$: if the sensor device at x_i^* $(i=1,2)$ is used, we set $\tilde d_i=1$, otherwise, $\tilde d_i=0$ with $\tilde d=\tilde d_1+\tilde d_2\geq1$. Then the measurement is given by

$$y(t)=\begin{cases}\left[z(x_1^*,t),z(x_2^*,t)\right]^{\mathrm{T}}, & \tilde d=2,\\ \tilde d_1 z(x_1^*,t)+\tilde d_2 z(x_2^*,t), & \tilde d=1.\end{cases}\tag{3}$$

For a function $g\in L^2(0,1)$, $v=F_\gamma(g):=(\gamma I-\partial_{xx})^{-1}g$ is the unique solution to the boundary value problem: $-v_{xx}+\gamma v=g$, $v(0)=v'(1)=0$. Similar to the proof of [22, Th. 8.22], we can obtain that F_γ is linear, continuous, and self-adjoint. We then present system (1) as

$$\begin{aligned} &z_t=z_{xx}-\rho z+\alpha\beta F_\gamma(z),\\ &z(0,t)=d_1\mathbf{u}_1(t),\ \ z_x(1,t)=d_2\mathbf{u}_2(t). \end{aligned}\tag{4}$$

Let $\psi_1(x)=\cos(\pi x)$ and $\psi_2(x)=-\frac{1}{\pi}\sin(\pi x)$, $x\in[0,1]$. Note that ψ_1 and ψ_2 satisfy

$$\begin{aligned} &\psi_1''(x)=-\mu\psi_1(x),\ \ \psi_1(0)=1,\ \ \psi_1'(1)=0,\\ &\psi_2''(x)=-\mu\psi_2(x),\ \ \psi_2(0)=0,\ \ \psi_2'(1)=1, \end{aligned}$$

where $\mu=\pi^2$. Following [23] for heat equation, we introduce the change of variables

$$\begin{aligned} &w(x,t)=z(x,t)-\psi(x)\mathbf{u}(t),\\ &\psi(x)=\begin{cases}[\psi_1(x),\psi_2(x)], & d=2,\\ d_1\psi_1(x)+d_2\psi_2(x), & d=1,\end{cases}\\ &\mathbf{u}(t)=\begin{cases}[\mathbf{u}_1(t),\mathbf{u}_2(t)]^{\mathrm{T}}, & d=2,\\ d_1\mathbf{u}_1(t)+d_2\mathbf{u}_2(t), & d=1.\end{cases} \end{aligned}\tag{5}$$

Combining (1) and (5), we obtain the equivalent system:

$$\begin{aligned} &w_t=w_{xx}-\rho w+\alpha\beta F_\gamma(w)-\psi[\Xi_0\mathbf{u}(t)+\dot{\mathbf{u}}(t)]\\ &\qquad+\alpha\beta\tilde F_\gamma(\psi)\mathbf{u}(t),\\ &w(0,t)=w_x(1,t)=0,\\ &w(x,0)=z_0(x),\ \ \ x\in(0,1), \end{aligned}\tag{6}$$

where $\Xi_0=\mathrm{diag}\{\mu+\rho,\mu+\rho\}$, $\tilde F_\gamma(\psi)=[F_\gamma(\psi_1),F_\gamma(\psi_2)]$ for $d=2$ and $\Xi_0=\mu+\rho$, $\tilde F_\gamma(\psi)=d_1F_\gamma(\psi_1)+d_2F_\gamma(\psi_2)$ for $d=1$. The measurement (3) becomes

$$y(t)=\begin{cases}\begin{bmatrix}w\big(x_1^*,t\big)+\psi\big(x_1^*\big)\mathbf{u}(t)\\ w\big(x_2^*,t\big)+\psi\big(x_2^*\big)\mathbf{u}(t)\end{bmatrix}, & \tilde d=2,\\ \sum_{i=1}^2\tilde d_i\big[w\big(x_i^*,t\big)+\psi\big(x_i^*\big)\mathbf{u}(t)\big], & \tilde d=1.\end{cases}\tag{7}$$

Henceforth we treat $\mathbf{u}(t)\in\mathbb{R}^d$ as an additional state variable, subject to the dynamics

$$\dot{\mathbf{u}}(t)=-\Xi_0\mathbf{u}(t)+\mathbf{v}(t),\ \ t>0,\ \ \ \mathbf{u}(0)=0,\tag{8}$$

whereas $\mathbf{v}(t)\in\mathbb{R}^d$ is the new control input. Then (6) can be presented as

$$\begin{aligned} &w_t=w_{xx}-\rho w+\alpha\beta F_\gamma(w)-\psi\mathbf{v}(t)+\alpha\beta\tilde F_\gamma(\psi)\mathbf{u}(t),\\ &w(0,t)=w_x(1,t)=0. \end{aligned}\tag{9}$$

B. Modal Decomposition and Observer/Controller Design

Define the operator:

$$\begin{aligned} &\mathcal{A}:\mathcal{A}w=\partial_{xx}w-\rho w+\alpha\beta F_\gamma(w),\\ &\mathcal{D}(\mathcal{A})=\Big\{w\in H^2(0,1),w(0)=w'(1)=0\Big\}. \end{aligned}\tag{10}$$

According to [9, Th. 1], the operator $\mathcal{A}$ generates an analytic C_0-semigroup on $L^2(0,1)$. From [9, Lemma 2] or [11, Th. 6], the eigenvalues and the corresponding eigenfunctions of $\mathcal{A}$ are given by

$$\begin{aligned} &\bar\lambda_n=-\rho-\lambda_n+\frac{\alpha\beta}{\lambda_n+\gamma},\ \ n\geq1,\\ &\phi_n(x)=\sqrt{2}\sin((n-0.5)\pi x),\ \ n=1,2,\ldots, \end{aligned}\tag{11}$$

where λ_n are defined below (2). The eigenfunctions form a complete orthonormal system in $L^2(0,1)$. Since $\lim_{n\to\infty}\bar{\lambda}_n = -\infty$, for given $\delta > 0$ we can find N_0 such that

$$\bar{\lambda}_n + \delta < 0, \quad n > N_0, \tag{12}$$

where N_0 will define the number of modes to be controlled. Let $N \geq N_0$, $N \in \mathbb{N}$ be the dimension of the observer.

The eigenvalues $\{\bar{\lambda}_n\}_{n=1}^{N_0}$ are simple if and only if α, β, γ satisfy $-\alpha\beta \neq (\lambda_{n_1}+\gamma)(\lambda_{n_2}+\gamma)$ for $1 \leq n_1 < n_2 \leq N_0$. Thus, for $\alpha\beta > 0$ and $\gamma > 0$ the eigenvalues are simple. Note that when $-\alpha\beta = (\lambda_{n_1}+\gamma)(\lambda_{n_2}+\gamma)$, multiple eigenvalues appear, making our single actuation inapplicable. In this case, the conditions in [9] are feasible for large ρ.

We claim that the maximum multiplicity of $\{\bar{\lambda}_n\}_{n=1}^{N_0}$ is 2. If there were 3 multiple eigenvalues: $\bar{\lambda}_{n_1} = \bar{\lambda}_{n_2} = \bar{\lambda}_{n_3}$, it would follow that $\lambda_{n_1} = \lambda_{n_2} = \frac{\alpha\beta}{\lambda_{n_3}+\gamma} - \gamma$, contradicting the fact that $\{\lambda_n\}_{n=1}^{\infty}$ are simple. We assume

$$d, \tilde{d} \geq \text{ maximum multiplicity of } \{\bar{\lambda}_n\}_{n=1}^{N_0}, \tag{13}$$

meaning that if there are multiple eigenvalues, we must have $d = \tilde{d} = 2$, and if the eigenvalues $\bar{\lambda}_n$ are simple (as assumed in [11] for state-feedback control), a single actuation (i.e., $d = 1$) and a single measurement (i.e., $\tilde{d} = 1$) are sufficient.

We present the solution to (9) as

$$w(\cdot,t) \overset{L^2}{=} \sum_{n=1}^{\infty} w_n(t)\phi_n, \;\; w_n(t) = \langle w(\cdot,t), \phi_n\rangle. \tag{14}$$

with $\{\phi_n\}_{n=1}^{\infty}$ defined in (11). Integration by parts implies

$$\langle F_\gamma(g), \phi_n\rangle = \frac{1}{\lambda_n+\gamma}\langle g, \phi_n\rangle, \;\; \forall g \in L^2(0,1). \tag{15}$$

Differentiating under the integral, integrating by parts and using (10), (11), (15), we obtain

$$\begin{aligned} &\dot{w}_n(t) = \bar{\lambda}_n w_n(t) - \frac{\alpha\beta}{\lambda_n+\gamma} b_n \mathbf{u}(t) + b_n \mathbf{v}(t), \;\; t > 0, \\ &w_n(0) = \langle w(\cdot,0), \phi_n\rangle, \\ &b_n = \begin{cases} \left[\frac{\phi_n'(0)}{\mu-\lambda_n}, \frac{\phi_n(1)}{\mu-\lambda_n}\right], d = 2, \\ \frac{d_1\phi_n'(0)}{\mu-\lambda_n} + \frac{d_2\phi_n(1)}{\mu-\lambda_n}, \;\; d = 1. \end{cases} \end{aligned} \tag{16}$$

Following [15] for heat equation, we construct the finite-dimensional observer

$$\hat{w}(x,t) = \sum_{n=1}^{N} \hat{w}_n(t)\phi_n(x), \tag{17}$$

with $\hat{w}_n(t)$, $1 \leq n \leq N$ satisfying

$$\begin{aligned} \dot{\hat{w}}_n(t) &= \bar{\lambda}_n \hat{w}_n(t) - \frac{\alpha\beta}{\lambda_n+\gamma} b_n \mathbf{u}(t) + b_n \mathbf{v}(t) \\ &\quad - l_n\big(\hat{y}(t) - y(t)\big), \;\; t > 0, \quad \hat{w}_n(0) = 0, \end{aligned} \tag{18}$$

where $y(t)$ is given by (7), $l_n \in \mathbb{R}^{1\times\tilde{d}}$, $n = 1, \ldots, N$ are observer gains, and

$$\hat{y}(t) = \begin{cases} \begin{bmatrix} \hat{w}(x_1^*, t) + \psi(x_1^*)\mathbf{u}(t) \\ \hat{w}(x_2^*, t) + \psi(x_2^*)\mathbf{u}(t) \end{bmatrix}, & \tilde{d} = 2, \\ \sum_{i=1}^{2} \tilde{d}_i\big[\hat{w}(x_i^*, t) + \psi(x_i^*)\mathbf{u}(t)\big], & \tilde{d} = 1. \end{cases}$$

Introduce the notations

$$c_n = \begin{cases} \big[\phi_n(x_1^*), \phi_n(x_2^*)\big]^{\mathrm{T}}, \;\; \tilde{d} = 2, \\ \tilde{d}_1\phi_n(x_1^*) + \tilde{d}_2\phi_n(x_2^*), \;\; \tilde{d} = 1, \end{cases} \; n \in \mathbb{N},$$

$$C_0 = \big[c_1, \ldots, c_{N_0}\big] \in \mathbb{R}^{\tilde{d}\times N_0}, \;\; A_0 = \mathrm{diag}\{\bar{\lambda}_n\}_{n=1}^{N_0}.$$

We choose $x_1^*, x_2^* \in (0,1]$ such that

$$\begin{aligned} &\mathrm{rank}(c_n) = 1, \;\text{ if } \bar{\lambda}_n \text{ is simple}, \; 1 \leq n \leq N_0, \\ &\mathrm{rank}\big[c_{n_1}, c_{n_2}\big] = 2, \;\text{ if } \bar{\lambda}_{n_1} = \bar{\lambda}_{n_2}, 1 \leq n_1 < n_2 \leq N_0. \end{aligned} \tag{19}$$

From [19, Lemma 1], we obtain that the pair (A_0, C_0) is observable. For $\delta > 0$, we choose $L_0 = [l_1^{\mathrm{T}}, \ldots, l_{N_0}^{\mathrm{T}}]^{\mathrm{T}}$ satisfying the following Lyapunov inequality

$$P_c(A_0 - L_0C_0) + (A_0 - L_0C_0)^{\mathrm{T}}P_c < -2\delta P_c, \tag{20}$$

for some $0 < P_c \in \mathbb{R}^{N_0\times N_0}$. Furthermore, as in [15], [16], [24], we choose $l_n = 0_{1\times\tilde{d}}$ for $n = N_0+1, \ldots, N$, to guarantee the LMI feasibility and derive reduced-order LMIs.

Define the following notations:

$$\Lambda_0 = \left[\frac{\alpha\beta}{\lambda_1+\gamma}b_1^{\mathrm{T}}, \ldots, \frac{\alpha\beta}{\lambda_{N_0}+\gamma}b_{N_0}^{\mathrm{T}}\right]^{\mathrm{T}}, \;\; B_0 = \big[b_1^{\mathrm{T}}, \ldots, b_{N_0}^{\mathrm{T}}\big]^{\mathrm{T}},$$

$$\tilde{A}_0 = \begin{bmatrix} -\Xi_0 & 0 \\ -\Lambda_0 & A_0 \end{bmatrix}, \;\; \tilde{B}_0 = \begin{bmatrix} I_2 \\ B_0 \end{bmatrix}.$$

Lemma 1: The pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable.

Proof: We focus exclusively on the case that there are multiple eigenvalues among the first N_0 eigenvalues, which together with (13) implies $d = 2$. The scenario where the eigenvalues are distinct can be proven in a similar manner.

By the Hautus criterion (see, e.g., [25, Proposition 1.5.5]), the controllability of $(\tilde{A}_0, \tilde{B}_0)$ is equivalent to $\mathrm{rank}[\tilde{A}_0 - \lambda I, \tilde{B}_0] = N_0 + 2$ for all $\lambda \in \sigma(\tilde{A}_0) = \{-(\mu+\rho), -(\mu+\rho), \bar{\lambda}_1, \ldots, \bar{\lambda}_{N_0}\}$. For $\lambda = -(\mu+\rho)$, we have

$$\begin{aligned} &\big[\tilde{A}_0 - \lambda I, \tilde{B}_0\big] \\ &= \begin{bmatrix} 0_{2\times2} & 0 & I_2 \\ -\mathrm{col}\left\{\frac{\alpha\beta b_n}{\lambda_n+\gamma}\right\}_{n=1}^{N_0} & \mathrm{diag}\big\{\bar{\lambda}_n + \mu + \rho\big\}_{n=1}^{N_0} & B_0 \end{bmatrix}. \end{aligned}$$

Since $\bar{\lambda}_n + \mu + \rho \neq 0$, $n = 1, \ldots, N_0$, it is clear that $\mathrm{rank}[\tilde{A}_0 - \lambda I, \tilde{B}_0] = N_0 + 2$. Let $\bar{\lambda}_{n_1}, \bar{\lambda}_{n_2}$ be the multiple eigenvalues. For $\lambda = \bar{\lambda}_{n_1} = \bar{\lambda}_{n_2}$, by elementary matrix transformations, we find that $[\tilde{A}_0 - \lambda I, \tilde{B}_0]$ is equivalent to

$$\left[\begin{array}{c|c|c|c} 0_{2\times2} & 0 & 0_{2\times2} & I_2 \\ \hline 0 & \Phi_{n_1,n_2} & 0 & \begin{matrix} b_{n_1} \\ b_{n_2} \end{matrix} \\ \hline \hat{\Xi}_2 & \hat{\Xi}_1 & 0 & \hat{B}_0 \end{array}\right], \tag{21}$$

where $\Phi_{n_1,n_2} = \begin{bmatrix} \phi_{n_1}'(0) & \phi_{n_1}(1) \\ \phi_{n_2}'(0) & \phi_{n_2}(1) \end{bmatrix}$, $\hat{B}_0 = \mathrm{col}\{b_n\}_{n=1, n\neq n_1,n_2}^{N_0}$, $\hat{\Xi}_1 = \mathrm{col}\{(\mu - \lambda_{n_1} + \frac{\alpha\beta}{\lambda_{n_1}+\gamma} - \frac{\alpha\beta}{\lambda_n+\gamma})b_n\}_{n=1,n\neq n_1,n_2}^{N_0}$, $\hat{\Xi}_2 = \mathrm{diag}\{\bar{\lambda}_n - \bar{\lambda}_{n_1}\}_{n=1,n\neq n_1,n_2}^{N_0}$. Since $\mathrm{rank}(\Phi_{n_1,n_2}) = 2$ and $\bar{\lambda}_n - \bar{\lambda}_{n_1} \neq 0$ for $n \neq n_1, n_2$, $1 \leq n \leq N_0$, we have $\mathrm{rank}[\tilde{A}_0 - \lambda I, \tilde{B}_0] = N_0 + 2$. ■

Remark 2: In [19], [20], [21] for high-dimensional heat equations, multiple control inputs on the same in-domain subset or boundary are introduced to address controllability issues arising from multiple eigenvalues. If we follow [19], [20], [21] and assume $z(0,t)=0$, $z_x(1,t)=\mathbf{u}_1(t)+\mathbf{u}_2(t)$, we find that the corresponding Φ_{n_1,n_2} in (21) becomes $\begin{bmatrix}\phi_{n_1}(1) & \phi_{n_1}(1)\\ \phi_{n_2}(1) & \phi_{n_2}(1)\end{bmatrix}$. In this case, $\text{rank}[\tilde{A}_0-\lambda I,\tilde{B}_0]=N_0+1$, meaning that $(\tilde{A}_0,\tilde{B}_0)$ is not controllable if $\bar{\lambda}_{n_1}=\bar{\lambda}_{n_2}$. This indicates that the method of [19], [20], [21] cannot be applied to address multiple eigenvalues in parabolic-elliptic equations.

Let $K_0\in\mathbb{R}^{d\times(N_0+d)}$ be the controller gain (it will be found from LMIs (28) and (27) below). We propose the following finite-dimensional controller:

$$\begin{aligned}\mathbf{v}(t)&=-K_0\hat{w}^{N_0}(t),\\ \hat{w}^{N_0}(t)&=\text{col}\{\mathbf{u}(t),\hat{w}_1(t),\ldots,\hat{w}_{N_0}(t)\}.\end{aligned}\tag{22}$$

Since $\mathcal{A}$ defined in (10) generates an analytic C_0-semigroup on $L^2(0,1)$ and $F_\gamma: L^2(0,1)\to L^2(0,1)$ is linear and continuous, the well-posedness of the closed-loop system (8), (9) with control input (22) can be established in a manner similar to[16, eqs. (2.15)–(2.18)]: given $z_0\in H^1(0,1)$, system (9) with control input (22) has a unique classical solution $w\in C([0,\infty),L^2(0,1))\cap C^1([0,\infty),L^2(0,1))$, and $w(\cdot,t)\in\mathcal{D}(\mathcal{A})$ for all $t>0$.

C. H^1 Exponential Stability

Let $e_n(t)=w_n(t)-\hat{w}_n(t)$. We have

$$\hat{y}(t)-y(t)=-\sum_{n=1}^{N}c_ne_n(t)-\zeta(t),\tag{23}$$

where $\zeta(t)=\sum_{n=N+1}^{\infty}c_nw_n(t)$. By using the Cauchy-Schwarz inequality, we obtain

$$|\zeta(t)|^2\le\kappa_N\sum_{n=N+1}^{\infty}\lambda_nw_n^2(t),\tag{24}$$

where $\kappa_N=\sum_{n=N+1}^{\infty}\frac{|c_n|^2}{\lambda_n}=\sum_{n=N+1}^{\infty}\frac{|c_n|^2}{(n-\frac{1}{2})^2\pi^2}$.

From (16), (18), and (23), we obtain

$$\dot{e}_n(t)=\bar{\lambda}_ne_n(t)-l_n\left[\sum_{n=1}^{N}c_ne_n(t)+\zeta(t)\right].\tag{25}$$

Denote by $e^{N_0}(t)=\text{col}\{e_n(t)\}_{n=1}^{N_0}$ and $e^{N-N_0}(t)=\text{col}\{e_n(t)\}_{n=N_0+1}^{N}$. Since $l_n=0$ for $N_0+1<n\le N$, we see that $\dot{e}^{N-N_0}(t)=A_1e^{N-N_0}(t)$, $t\ge0$, where $A_1=\text{diag}\{\bar{\lambda}_n\}_{n=N_0+1}^{N}$ is a stable matrix due to (12). By (18), (22), (25), we obtain

$$\begin{aligned}\dot{\hat{w}}^{N_0}(t)&=\left(\tilde{A}_0-\tilde{B}_0K_0\right)\hat{w}^{N_0}(t)+\tilde{L}_0C_0e^{N_0}(t)\\&\quad+\tilde{L}_0\zeta(t)+\tilde{L}_0C_1e^{N-N_0}(t),\end{aligned}\tag{26a}$$

$$\dot{e}^{N_0}(t)=(A_0-L_0C_0)e^{N_0}(t)-L_0[\zeta(t)+C_1e^{N-N_0}(t)],\tag{26b}$$

$$\dot{w}_n(t)=\bar{\lambda}_nw_n(t)-\left[\frac{\alpha\beta b_n\mathbf{1}_0}{\lambda_n+\gamma}+b_nK_0\right]\hat{w}^{N_0}(t),\ n>N,\tag{26c}$$

where $C_1=[c_{N_0+1},\ldots,c_N]$, $\tilde{L}_0=[0_{\tilde{d}\times d},L_0^{\mathrm{T}}]^{\mathrm{T}}$, $\mathbf{1}_0=[I_d,0_{d\times N_0}]$. Note that $\zeta(t)$ defined below (23) does not depend on $\{\hat{w}_n(t)\}_{n=N_0+1}^{N}$ which are exponentially decaying provided $\hat{w}^{N_0}(t)$ is exponentially decaying. Therefore, for the stability of (9) with control (22), it is sufficient to show the stability of the reduced-order closed-loop system (26).

Next, we derive LMI conditions for finding observer dimension N and gain K_0 ensuring the H^1 exponential stability of system (26). The main result is presented below.

Theorem 1: Consider system (8), (9) with $\alpha\beta\neq0$, $\rho\in\mathbb{R}$, $\gamma\neq-(n-0.5)^2\pi^2$, $n\in\mathbb{N}$, initial value $z_0\in H^1(0,1)$, measurement (7) with $x_1^*,x_2^*\in(0,1]$ satisfying (19) and control law (22). For decay rate $\delta>0$, let $N_0\in\mathbb{N}$ satisfy (12), $N\ge N_0$, and $d,\tilde{d}$ satisfy (13). Let there exist scalars $a_1,a_2,a_3>0$, matrices $0<\bar{P}_w\in\mathbb{R}^{(N_0+d)\times(N_0+d)}$, $0<P_e\in\mathbb{R}^{N_0\times N_0}$, $Y\in\mathbb{R}^{d\times(N_0+d)}$ such that the following LMIs are feasible:

$$\bar{\lambda}_{N+1}+\delta+\frac{a_1\lambda_{N+1}}{2}+\frac{a_2|\alpha\beta|\lambda_{N+1}}{2(\lambda_{N+1}+\gamma)^2}+\frac{a_3\kappa_N}{2}<0,\tag{27}$$

and

$$\begin{aligned}\hat{\Theta}_0&=\left[\begin{array}{ccc|cc}\hat{\Theta}_{11} & \tilde{L}_0C_0 & \tilde{L}_0 & Y^{\mathrm{T}} & \bar{P}_w\mathbf{1}_0^{\mathrm{T}}\\ * & \Theta_{22} & -P_eL_0 & 0 & 0\\ * & * & -a_3I & 0 & 0\\ \hline * & * & * & -\frac{a_1}{\rho_N}I & 0\\ * & * & * & * & -\frac{a_2}{|\alpha\beta|\rho_N}I\end{array}\right]<0,\\ \hat{\Theta}_{11}&=\tilde{A}_0\bar{P}_w+\bar{P}_w\tilde{A}_0^{\mathrm{T}}-\tilde{B}_0Y-Y^{\mathrm{T}}\tilde{B}_0+2\delta\bar{P}_w,\\ \Theta_{22}&=P_e(A_0-L_0C_0)+(A_0-L_0C_0)^{\mathrm{T}}P_e+2\delta P_e.\end{aligned}\tag{28}$$

Then solution $\mathbf{u}(t)$, $w(x,t)$ to (8), (9) under the control law (22), (18) with controller gain $K_0=Y\bar{P}_w^{-1}$ and the corresponding observer $\hat{w}(x,t)$ given by (17) satisfy

$$\begin{aligned}&|\mathbf{u}(t)|^2+\|w(\cdot,t)\|_{H^1}^2+\|w(\cdot,t)-\hat{w}(\cdot,t)\|_{H^1}^2\\&\le M\mathrm{e}^{-\delta t}\|z_0\|_{H^1}^2,\quad t\ge0,\end{aligned}\tag{29}$$

for some $M\ge1$. Moreover, LMIs (27) and (28) are always feasible for large enough N, and the feasibility of the LMIs with some N implies their feasibility for $N+1$.

Proof: We consider the Lyapunov function

$$\begin{aligned}V(t)&=\left|\hat{w}^{N_0}(t)\right|_{P_w}^2+\left|e^{N_0}(t)\right|_{P_e}^2\\&\quad+p_e\left|e^{N-N_0}(t)\right|^2+\sum_{n=N+1}^{\infty}\lambda_nw_n^2(t),\end{aligned}\tag{30}$$

where $P_w=\bar{P}_w^{-1}$ and $0<p_e\in\mathbb{R}$. Differentiation of $V(t)$ along (26) gives

$$\begin{aligned}\dot{V}(t)+2\delta V(t)&=\left(\hat{w}^{N_0}(t)\right)^{\mathrm{T}}\Big[P_w\left(\tilde{A}_0-\tilde{B}_0K_0\right)\\&\quad+\left(\tilde{A}_0-\tilde{B}_0K_0\right)^{\mathrm{T}}P_w+2\delta P_w\Big]\hat{w}^{N_0}(t)\\&\quad+2\left(\hat{w}^{N_0}(t)\right)^{\mathrm{T}}P_w\tilde{L}_0\left[C_0e^{N_0}(t)+\zeta(t)+C_1e^{N-N_0}(t)\right]\\&\quad+\left(e^{N_0}(t)\right)^{\mathrm{T}}\left[P_e(A_0-L_0C_0)+(A_0-L_0C_0)^{\mathrm{T}}P_e\right.\\&\quad\left.+2\delta P_e\right]e^{N_0}(t)-2\left(e^{N_0}(t)\right)^{\mathrm{T}}P_eL_0\left[\zeta(t)+C_1e^{N-N_0}(t)\right]\\&\quad+\sum_{n=N+1}^{\infty}2\lambda_n\left(\bar{\lambda}_n+\delta\right)w_n^2(t)\\&\quad-\sum_{n=N+1}^{\infty}2\lambda_nw_n(t)\left[\frac{\alpha\beta b_n}{\lambda_n+\gamma}\mathbf{1}_0\hat{w}^{N_0}(t)+b_nK_0\hat{w}^{N_0}(t)\right]\\&\quad+2p_e\left(e^{N-N_0}(t)\right)^{\mathrm{T}}(A_1+\delta I)\ e^{N-N_0}(t).\end{aligned}\tag{31}$$

By Young's inequality, we have for $a_1,a_2>0$,

$$-\sum_{n=N+1}^{\infty}2\lambda_nw_n(t)b_nK_0\hat{w}^{N_0}(t)$$

$$\le \sum_{n=N+1}^{\infty} \lambda_n^2 a_1 w_n^2(t) + \frac{\rho_N}{a_1}\left|K_0\hat{w}^{N_0}(t)\right|^2,$$
$$-\sum_{n=N+1}^{\infty} 2\lambda_n w_n(t)\frac{\alpha\beta}{\lambda_n+\gamma} b_n \mathbf{1}_0 \hat{w}^{N_0}(t)$$
$$\le \sum_{n=N+1}^{\infty} \frac{|\alpha\beta| a_2}{(\lambda_n+\gamma)^2}\lambda_n^2 w_n^2(t) + \frac{|\alpha\beta|\rho_N}{a_2}\left|\mathbf{1}_0\hat{w}^{N_0}(t)\right|^2, \quad (32)$$

where $\rho_N = \sum_{n=N+1}^{\infty}|b_n|^2 = \sum_{n=N+1}^{\infty}\frac{2(d_1\lambda_n+d_2)}{(\lambda_n-\mu)^2}$. Let $\eta(t) = \mathrm{col}\{\hat{w}^{N_0}(t), e^{N_0}(t), \zeta(t), e^{N-N_0}(t)\}$. Substituting (32) into (31), using (24) and S-procedure for $a_3 > 0$, we obtain

$$\dot{V}(t) + 2\delta V(t) + a_3\left[\kappa_N \sum_{n=N+1}^{\infty}\lambda_n w_n^2(t) - |\zeta(t)|^2\right]$$
$$\le \eta^{\mathrm{T}}(t)\Theta\eta(t) + \sum_{n=N+1}^{\infty} 2\lambda_n \Phi_n w_n^2(t), \quad (33)$$

where $\Phi_n = \bar{\lambda}_n + \delta + \frac{a_1\lambda_n}{2} + \frac{|\alpha\beta|a_2\lambda_n}{2(\lambda_n+\gamma)^2} + \frac{a_3\kappa_N}{2}$, $n > N$, and

$$\Theta = \begin{bmatrix} \Theta_0 & \Psi \\ * & 2p_e(A_1+2\delta I)\end{bmatrix}, \Psi = \begin{bmatrix} P_w\tilde{L}_0C_1 \\ -P_eL_0C_1 \\ 0\end{bmatrix},$$
$$\Theta_0 = \left[\begin{array}{c|c|c} \Theta_{11} & P_w\tilde{L}_0C_0 & P_w\tilde{L}_0 \\ \hline * & \Theta_{22} & -P_eL_0 \\ \hline * & * & -a_3 I\end{array}\right]. \quad (34)$$

Here $\Theta_{11} = P_w(\tilde{A}_0 - \tilde{B}_0K_0) + (\tilde{A}_0 - \tilde{B}_0K_0)^{\mathrm{T}}P_w + 2\delta P_w + \frac{\rho_N}{a_1}K_0^{\mathrm{T}}K_0 + \frac{|\alpha\beta|\rho_N}{a_2}\mathbf{1}_0^{\mathrm{T}}\mathbf{1}_0$. Since $A_1 + \delta I < 0$ due to (12), by Schur complement for $p_e \to \infty$, we obtain that $\Theta < 0$ iff $\Theta_0 < 0$. By multiplying Θ_0 in (34) by $\mathrm{diag}\{P_w^{-1}, I, I\}$ from the right and the left, using Schur complement, and recalling $\bar{P}_w = P_w^{-1}$, we obtain that (28) implies $\Theta_0 < 0$. Moreover, since $\bar{\lambda}_n \to -\infty$ and is monotonically for $n > N$, we have $\Phi_n < 0$ for all $n > N$ if (27) holds. If LMIs (27) and (28) are feasible, the controller gain is obtained by $K_0 = Y\bar{P}_w^{-1}$. In this case, we obtain from (33) that $\dot{V}(t) + 2\delta V(t) < 0$, which together with (30) implies (29) for some $M \ge 1$.

We show next that for any system parameters $\alpha\beta \neq 0$, $\gamma \neq -\lambda_n$, $\rho \in \mathbb{R}$, LMIs (27) and (28) are always feasible for large enough N. To show the feasibility, we choose $a_1 = a_2 = 1$ and $a_3 = N$. Since the pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable (see Lemma 1), we take K_0 such that $\tilde{A}_0 - \tilde{B}_0K_0 + \delta I$ is Hurwitz. Then we can find P_w that solves the following Lyapunov equality:

$$P_w\left(\tilde{A}_0 - \tilde{B}_0K_0 + \delta I\right) + \left(\tilde{A}_0 - \tilde{B}_0K_0 + \delta I\right)^{\mathrm{T}}P_w = -I. \quad (35)$$

Since (A_0, C_0) is observable, we take L_0 such that $A_0 - L_0C_0 + \delta I$ is Hurwitz. For $\chi > 0$ satisfying $\chi I - P_w\tilde{L}_0C_0C_0^{\mathrm{T}}\tilde{L}^{\mathrm{T}}P_w > 0$, we can find P_e that solves the following Lyapunov equality:

$$P_e(A_0 - L_0K_0 + \delta I) + (A_0 - L_0C_0 + \delta I)^{\mathrm{T}}P_e = -\chi I. \quad (36)$$

From the definitions of ρ_N below (32) and κ_N below (24), we have $\rho_N = O(N^{-1})$, $\kappa_N = O(N^{-1})$, $N \to \infty$. Recalling $a_1 = a_2 = 1$, $a_3 = N$, $\bar{\lambda}_n$ in (11), and λ_n below (12), we find that (27) holds for $N \to \infty$. Substituting (35), (36), and $a_3 = N$ into Θ_0 defined in (34), we find that $\Theta_0 < 0$ is always feasible for large enough N.

Note that the size of $\hat{\Theta}_0$ does not change when N grows. Moreover, both ρ_N and κ_N monotonically decrease with N. Therefore, if LMIs (28) and (27) are feasible for some N, then they are feasible for $N+1$. ■

Corollary 1: Under the conditions of Theorem 1, the following estimates hold for $z(x,t)$ and $v(x,t)$ (state of (1)):

$$\|z(\cdot,t)\|_{H^1}^2 + \|v(\cdot,t)\|_{H^1}^2 \le \bar{M}\mathrm{e}^{-\delta t}\|z_0\|_{H^1}, t \ge 0, \bar{M} \ge 1. \quad (37)$$

Proof: First, for $v(\cdot,t)$, we have the following estimate:

$$\|v(\cdot,t)\|_{H^1}^2 = \sum_{n=1}^{\infty}\lambda_n\langle v(\cdot,t),\phi_n\rangle^2 = \sum_{n=1}^{\infty}\lambda_n\langle F_\gamma(z(\cdot,t)),\phi_n\rangle^2$$
$$\overset{(15)}{=} \sum_{n=1}^{\infty}\frac{\lambda_n}{\lambda_n+\gamma}\langle z(\cdot,t),\phi_n\rangle^2 \le \bar{M}_0\|z(\cdot,t)\|_{H^1}^2, \quad (38)$$

where $\bar{M}_0 = \sup_{n\ge 1}\frac{1}{|\lambda_n+\gamma|}$. From (5) and (29), we have

$$\|z(\cdot,t)\|_{H^1}^2 \le 2\|w(\cdot,t)\|_{H^1}^2 + 2\|\psi\|_{H^1}^2|\mathbf{u}(t)|^2$$
$$\le \bar{M}_1\mathrm{e}^{-\delta t}\|z_0\|_{H^1}, \quad t \ge 0,$$

for some $\bar{M}_1 \ge 1$, which together with (38) implies (37). ■

Remark 3: For the case of simple eigenvalues, single actuation and single measurement are sufficient, and our method is effective for Neumann boundary conditions as in [9] and Dirichlet boundary conditions as in [11]. Especially, for the following actuation and measurement as in [9]:

$$v_x(0,t) = v_x(1,t) = z_x(0,t) = 0, \; z_x(1,t) = \mathbf{u}(t),$$
$$y(t) = z(1,t), \quad (39)$$

the corresponding ψ, μ, λ_n, ϕ_n are replaced by

$$\psi(x) = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right), \mu = \frac{\pi^2}{4}, \lambda_n = (n-1)^2\pi^2,$$
$$\phi_1 \equiv 1, \; \phi_n(x) = \sqrt{2}\cos((n-1)\pi x), \quad n \ge 2. \quad (40)$$

For the stability analysis, we consider the Lyapunov function (30) with $\sum_{n=N+1}^{\infty}\lambda_n w_n^2(t)$ replaced by $\sum_{n=N+1}^{\infty}(\lambda_n+1)w_n^2(t)$. This is because for the Neumann boundary conditions, $\|w\|_{H^1}^2$ is equivalent to $\sum_{n=N+1}^{\infty}(\lambda_n+1)w_n^2$. By arguments similar to (31)-(27), we can obtain the exponential stability if (28) and the following LMI are feasible:

$$\bar{\lambda}_{N+1} + \delta + \frac{a_1}{2} + \frac{a_2|\alpha\beta|}{2(\lambda_{N+1}+\gamma)^2} + \frac{\beta\kappa_N}{2} < 0, \quad (41)$$

where κ_N, ρ_N are replaced by $\kappa_N = \sum_{n=N+1}^{\infty}\frac{|c_n|^2}{\lambda_n+1}$, $\rho_N = \sum_{n=N+1}^{\infty}(\lambda_n+1)b_n^2 = \sum_{n=N+1}^{\infty}\frac{2(\lambda_n+1)}{(\lambda_n-\pi^2/4)^2}$. The LMIs (28) and (41) are always feasible for any decay rate and system parameters provided $-\alpha\beta \neq (\lambda_{n_1}+\gamma)(\lambda_{n_2}+\gamma)$ for $1 \le n_1 < n_2 \le N_0$.

Remark 4: The modal decomposition method in this letter presents complementary results to [9] based on the backstepping: it allows a finite-dimensional observer-based control without any approximations, and can achieve any desired decay rate. Besides, the modal decomposition method is easily applicable to in-domain control/observer, and is more robust with respect to delays (see [26, Remark 2] and [19]).

TABLE I
MINIMUM N FROM LMIS (28) AND (41) FOR $\delta = 1$

ρ	-20	-15	-8	-2	0
N_0 (from (12))	2	2	1	1	1
L_0 (from (20))	$[206.7, 93.3]^{\mathrm{T}}$	$[75.9, 24.4]^{\mathrm{T}}$	22.8	7.9	3.6
min N	10	6	3	2	1

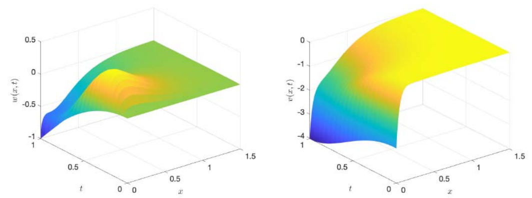

Fig. 1. Evolution of the state $w(x, t)$ (left) and $v(x, t)$ (right) with parameters (42) and gains (43), (44).

III. NUMERICAL EXAMPLES

We first consider system (1) with Neumann boundary conditions (39) and the positive parameters $\gamma = 1$, $\alpha = 0.5$, $\beta = 1$, where the eigenvalues are simple. We consider the measurement $y(t) = z(1, t)$. Note that [9, Th. 10] (for state feedback) is not applicable for $\rho < 0.0615$, whereas for $\rho > 0.5$ the open-loop system is stable and the decay rate can be slightly improved by the feedback. Our LMI conditions (28), (41) (with ψ, μ, λ_n, ϕ_n in (40) and κ_N, ρ_N above (41)) are feasible for any $\rho \in \mathbb{R}$ and decay rate $\delta > 0$ for appropriate N_0 and N. Setting $\delta = 1$, we choose $\rho \in \{-20, -15, -8, -2, 0\}$, determine N_0 from (12), and derive the observer gain L_0 from (20). The LMIs (28), (41) were verified to find minimal N that preserves feasibility. The results are given in Table I.

We show that for the case of multiple eigenvalues our bilateral control works efficiently. Consider (1) with boundary conditions (2) and the following parameters:

$$\alpha = -\pi^2/2, \;\; \beta = 5\pi^2/2, \;\; \gamma = \pi^2/2, \;\; \rho = -3\pi^2. \quad (42)$$

We have $\bar{\lambda}_1 = \bar{\lambda}_2 > 0$ and $\bar{\lambda}_n < 0$ for $n \geq 3$, which results in an unstable open-loop system. Let $d = \tilde{d} = 2$. Take the decay rate $\delta = 1$. From (12) we find $N_0 = 2$. Choosing $x_1^* = \frac{1}{2}$ and $x_2^* = 1$, we see that (19) is satisfied and $|c_n|^2 = 3$, $n = 1, 2, \ldots$. From (20), we obtain

$$L_0 = \begin{bmatrix} 3.2189 & 1.9622 \\ 3.2189 & -1.9622 \end{bmatrix}. \quad (43)$$

The LMIs (27) and (28) in Theorem 1 were verified to be feasible for $N = 3$. We obtain the controller gain

$$K_0 = \begin{bmatrix} 45.054 & -36.091 & 225.378 & 25.052 \\ -393.943 & -16.054 & 881.609 & -257.704 \end{bmatrix}. \quad (44)$$

For the simulation of the closed-loop system, we take the initial condition $z_0(x) = x^2 - 2x$. The simulation was carried out by using the FTCS method with time step 0.001 and space step 0.05. The simulation results are presented in Fig. 1 and confirm the theoretical analysis.

We next show that bilateral control for the simple eigenvalue case reduces the observer dimension. If we select $\gamma = \pi^2/2 + 0.01$, we have simple eigenvalues and obtain observer gain $L_0 = \begin{bmatrix} 3.2588 & 1.9934 \\ 3.2248 & -1.9591 \end{bmatrix}$ from (20). The LMIs (27) and (28) in Theorem 1 were verified to be feasible for bilateral actuations ($d_1 = d_2 = 1$) with minimal $N = 3$, for single Neumann actuation ($d_1 = 0, d_2 = 1$) with minimal $N = 7$, and for single Dirichlet actuation ($d_1 = 1, d_2 = 0$) with minimal $N = 26$.

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