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Averaging-based stability of discrete-time delayed systems via a novel delay-free transformation*

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***Abstract*—In this paper, we study, for the first time, the stability of linear delayed discrete-time systems with small parameter $\varepsilon > 0$ and rapidly-varying coefficients. Recently, an efficient constructive approach to averaging-based stability via a novel delay-free transformation was introduced for continuous-time systems. Our paper extends this approach to discrete-time systems. We start by introducing a discrete-time change of variables that leads to a perturbed averaged system. By employing Lyapunov analysis, we derive Linear Matrix Inequalities (LMIs) for finding the maximum values of the small parameter $\varepsilon > 0$ and delay (either constant or time-varying) that guarantee exponential stability of the original system. We show that differently from the continuous-time, in the discrete-time, given any bounded delay, there exists a small enough ε such that our LMIs are feasible (i.e. the system is exponentially stable). Numerical examples illustrate the efficiency of the proposed approach.**

I. Introduction

Averaging is considered as one of the most efficient methods to deal with the stability of control systems with rapidly time-varying almost periodic coefficients depending on a small parameter $\varepsilon > 0$ [2], [10], [13]. Different features of these systems has been widely studied by the control community [7], [10], [11], [12], mainly for their modern engineering applications [1], [4], [15], [17]. The key idea behind the asymptotic averaging method is that the asymptotic stability of the original rapidly-varying system is guaranteed for small enough values of the parameter ε if the averaged system is exponentially stable. However, a well-known drawback of the classical averaging method is the lack of an efficient quantitative upper bound on ε that preserves the stability of the original system.

Recently, a novel constructive time-delay approach to periodic averaging of continuous-time systems has been presented in [6]. By backward integration of the original system, the resulting system is presented as a time-delay (neutral type) system with time-delays of the length of the small parameter $\varepsilon > 0$. The stability of the resulting system was shown to imply the stability of the original system [6]. Then, direct Lyapunov-Krasovkii method was applied to obtain LMI conditions which provide an efficient upper bound on the small parameter ε that guarantees the stability of the original system provided the corresponding averaged system is exponentially stable. Extensions of this time-delay approach for input-to-state stability (ISS) and L_2-gain analysis of systems with constant/time-varying delays were presented in [3], [6], [19]. Moreover, it was extended to ISS analysis of perturbed discrete-time systems [18].

In the recent paper [9], a novel constructive approach for linear continuous-time systems with rapidly-varying almost periodic coefficients was introduced. Differently from the time-delay approach, the method of [9] relies on a novel non-delayed transformation which yields simpler analysis and reduces the conservatism on the upper bounds of the system parameters. This approach was applied to averaging of systems with both constant and time-varying delays. For the discrete-time systems in the presence of delays, constructive discrete-time results are still missing.

Note that there are no existing results (even qualitative) on averaging of discrete-time systems in the presence of delays. In this paper, we fill this gap by extending the approach of [9] to linear discrete-time systems, including systems with constant or time-varying delays. Although the fundamental ideas are inspired by the results of the continuous case [9], construction of the appropriate transformations and the subsequent Lyapunov analysis are not an immediate extension from the continuous framework, but rather require significant adaptation to the discrete-time case. Differently from [18], we consider a new presentation of the linear discrete-time system, where the system matrix is presented as a linear combination of a Hurwitz matrix and constant matrices multiplied by *scalar* rapidly-varying terms with zero average. We introduce a new discrete-time transformation of the rapidly-varying coefficients. Then, by using Lyapunov analysis, we obtain explicit LMI conditions which guarantee exponential stability of the target system (and eventually the original system). Moreover, we show that the feasibility of the LMIs is guaranteed for small enough values of the system parameters. Furthermore, differently from the continuous-time, given *any* bounded delay, there exist small enough ε such that the LMIs are feasible (i.e. the system is exponentially stable). Numerical examples demonstrate the efficiency of the suggested method. A conference version of the paper, confined to consideration of non-delayed systems and systems with constant delays, will be presented at ECC 2024 [8].

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $|\cdot|_1$ is ℓ^1 norm, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $||\cdot||$, 0_n and I_n are the zero matrix and the identity matrix of order n, respectively. $\mathbb{Z}_+$ is the set of non-negative integers. The notation $P>0$ for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The subdiagonal elements of a symmetric matrix are denoted by $*$, the superscript T denotes matrix transposition, and $\otimes$ denotes the Kronecker product. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2 = x^TPx$. For two integers p and q with $p\le q$, the notation $\mathbf{I}[p,q]$ refers to the set $\{p,p+1,...,q\}$ and we denote $|w|_{[p,q]} = \max_{s\in\mathbf{I}[p,q]}|w_s|$.

In the stability analysis below we will use the following:

Lemma 1. *(Jensen's inequality [5, Chapter 6]) For all $k\in\mathbb{Z}_+$ the following inequality holds:*

$$\frac{1}{d}|\sum_{i=k-d}^{k-1}[x_{i+1}-x_i]|_R^2 \le \sum_{i=k-d}^{k-1}|x_{i+1}-x_i|_R^2. \qquad (1)$$

Lemma 2. *(Reciprocally convex combination, [14], [5]) Given $R>0$, for any $G\in\mathbb{R}^{n\times n}$ such that the following inequality*

$$\begin{bmatrix} R & G\\ * & R\end{bmatrix}\ge 0, \qquad (2)$$

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holds. Then,

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geq \begin{bmatrix} R & G \\ * & R \end{bmatrix}, \ \forall \alpha \in (0,1). \tag{3}$$

II. Stability analysis via averaging of discrete-time systems

Consider the discrete-time system:

$$x_{k+1} = [I + \varepsilon A(k)]x_k, \ k \in \mathbb{Z}_+, \tag{4}$$

where $x_k \in \mathbb{R}^n$, $A(k) : \mathbb{Z}_+ \to \mathbb{R}^{n\times n}$, $\varepsilon > 0$ is a small parameter. We make the following assumption:

Assumption 1. *The matrices $A(k), \ k \in \mathbb{Z}_+$ satisfy*

$$A(k) = A_{av} + \sum_{i=1}^{N} a_i(k)A_i, \tag{5}$$

where A_{av} is a Hurwitz matrix and $\{a_i(k)\}_{i=1}^N, \ k \in \mathbb{Z}_+$ are T-periodic with zero average i.e.

$$\frac{1}{T}\sum_{j=k}^{k+T-1} a_i(j) = 0, \quad \forall k \geq 0, \quad \forall i \in \{1,2,...,N\}. \tag{6}$$

Remark 1. *Every matrix $A(k)$ can be presented as a linear combination in the form* (5)*. Moreover, assuming that the averages of a_i are zero poses no loss of generality, as we can subtract the averages from the corresponding functions and modify the matrix A_{av}.*

Remark 2. *For simplicity only we consider T-periodic a_i. Our method is applicable to almost periodic a_i, which satisfy*

$$\frac{1}{T}\sum_{j=k}^{k+T-1} a_i(j) = \Delta a_i(k), \quad \sup_{k\in\mathbb{Z}_+} ||\Delta a_i(k)|| \leq \Delta_{a_i}$$

with small enough Δ_{a_i}. The approach is also extensible to ISS of perturbed time-varying systems.

Using Assumption 1, we present (4) as

$$x_{k+1} - x_k = \varepsilon(A_{av} + \sum_{i=1}^{N} a_i(k)A_i)x_k. \tag{7}$$

In this paper, we will provide constructive LMI conditions for finding an upper bound on ε that guarantees the exponential stability of (4). We will derive LMIs by using two steps - system transformation and Lyapunov analysis.

Step I: System Transformation. For each $j \in \{1,2,...,N\}$, let

$$\rho_j(k) := -\frac{\varepsilon}{T}\sum_{i=k}^{k+T-1}(k+T-i)a_j(i). \tag{8}$$

Since a_i is a T-periodic function (whence bounded) by Assumption 1, one has $\rho_j = O(\varepsilon)$. Taking into account (6) and (8), we obtain

$$\begin{array}{r}\rho_j(k+1) - \rho_j(k) = -\frac{\varepsilon}{T}\sum_{i=k+1}^{k+T}(k+1+T-i)a_j(i) \\ +\frac{\varepsilon}{T}\sum_{i=k}^{k+T-1}(k+T-i)a_j(i) = \varepsilon a_j(k).\end{array} \tag{9}$$

Introduce the change of variables

$$z_k = x_k - \sum_{j=1}^{N}\rho_j(k)A_j x_k. \tag{10}$$

For simplicity of presentation we will proceed with the case $N = 2$. The general case follows the same arguments. Only minor modifications are required, which are related to the dimensions of the matrices. This is similar to the continuous case (see Remark 2.7 in [9]).

Clearly, for small enough ε, the matrix $I_n - \sum_{i=1}^{2}\rho(k)A_i$ is invertible, whence the transformation (10) is also invertible. A sufficient condition for this is given by the following inequality:

$$\delta_2 := \frac{1}{2}\sum_{i=1}^{2}\varepsilon T a_{i,M}||A_i|| < 1 \tag{11}$$

where $a_{i,M} := \sup_{k\in\mathbb{Z}}|a_i(k)|$, $i = 1,2$. Indeed, we have

$$\sup_{k\in\mathbb{Z}_+} ||\sum_{i=1}^{2}\rho_i(k)A_i|| \leq \delta_2 < 1, \tag{12}$$

and by employing a Neumann series and (12), we obtain

$$\sup_{k\in\mathbb{Z}_+}\left\| \left(I_n - \sum_{i=1}^{2}\rho(k)A_i\right)^{-1}\right\| \leq \delta_1 = (1-\delta_2)^{-1}. \tag{13}$$

So, under inequality (11), we have

$$|x_k| \leq \delta_1|z_k|, \ \forall k \in \mathbb{Z}_+, \tag{14}$$

$$|z_k| \leq (1+\delta_2)|x_k|, \ \forall k \in \mathbb{Z}_+. \tag{15}$$

Using equations (7), (9) and (10), we obtain

$$\begin{array}{r}z_{k+1} - z_k = \varepsilon A_{av}z_k - \varepsilon\sum_{j=1}^{2}\sum_{i=1}^{2}A_jA_i\rho_j(k+1)a_i(k)x_k \\ +\varepsilon\sum_{j=1}^{2}\rho_j(k)A_{av}A_jx_k - \varepsilon\sum_{j=1}^{2}A_jA_{av}\rho_j(k+1)x_k.\end{array} \tag{16}$$

Denoting

$$\mathcal{A} = [A_1, A_2], \ \mathcal{A}_1 = [A_1A_1, A_1A_2, A_2A_1, A_2A_2], \tag{17}$$

$$\begin{array}{l}\mathcal{Y}_\rho^{(m)}(k) = col\{\rho_j(k+m-1)x_k\}_{j=1}^2, \ m = 1,2, \\ \mathcal{Y}_{\rho,a}(k) = col\{\rho_1(k+1)a_1(k)x_k, \rho_1(k+1)a_2(k)x_k, \\ \qquad \rho_2(k+1)a_1(k)x_k, \rho_2(k+1)a_2(k)x_k\},\end{array} \tag{18}$$

the transformation (10) and system (16) can be presented as

$$z_k = x_k - \mathcal{A}\mathcal{Y}_\rho^{(1)}(k), \tag{19}$$

$$\begin{array}{r}z_{k+1} - z_k = \varepsilon A_{av}z_k + \varepsilon A_{av}\mathcal{A}\mathcal{Y}_\rho^{(1)}(k) \\ -\varepsilon\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}_\rho^{(2)}(k) - \varepsilon\mathcal{A}_1\mathcal{Y}_{\rho,a}(k).\end{array} \tag{20}$$

Note that since $\rho_j = O(\varepsilon)$, equation (20) is of the form

$$z_{k+1} - z_k = \varepsilon A_{av}z_k + O(\varepsilon^2).$$

Let

$$\begin{array}{l}H_\rho = col\{h_\rho^{(1)}, h_\rho^{(2)}\}, \\ H_{\rho,a} = col\{h_{\rho,a}^{(1,1)}, h_{\rho,a}^{(1,2)}, h_{\rho,a}^{(2,1)}, h_{\rho,a}^{(2,2)}\},\end{array} \tag{21}$$

where $h_\rho^{(i)}, \ h_{\rho,a}^{(i,j)}, \ i,j = 1,2$ are bounds such that

$$\rho_i^2(k) \leq h_\rho^{(i)}, \ \rho_i^2(k+1)a_j^2(k) \leq h_{\rho,a}^{(i,j)}, \ i,j = 1,2. \tag{22}$$

Since ρ_i and a_j are scalar functions, so are the upper bounds in (22). Since $\rho_j(k) = O(\varepsilon)$, one has in (21): $|H_\rho|_1 = O(\varepsilon^2)$ and $|H_{\rho,a}|_1 = O(\varepsilon^2)$. Then, for any positive diagonal matrices $\Lambda_\rho^{(m)} \in \mathbb{R}^{2\times 2}$, $m = 1,2$, $\Lambda_{\rho,a} \in \mathbb{R}^{4\times 4}$ the following are obtained from (22):

$$\begin{array}{l}\mathcal{Y}_\rho^{(m)}(k)^T(\Lambda_\rho^{(m)}\otimes I_n)\mathcal{Y}_\rho^{(m)}(k) \leq |\Lambda_\rho^{(m)}H_\rho|_1|x_k|^2, \\ \mathcal{Y}_{\rho,a}^T(k)(\Lambda_{\rho,a}\otimes I_n)\mathcal{Y}_{\rho,a}(k) \leq |\Lambda_{\rho,a}H_{\rho,a}|_1|x_k|^2.\end{array} \tag{23}$$

The matrices $\Lambda_\rho^{(1)}, \Lambda_\rho^{(2)}$ and $\Lambda_{\rho,a}$ will be decision variables in the LMIs derived below (see (33), (34)).

Step II: Lyapunov Analysis. For stability analysis of (20) subject to (19), we introduce the Lyapunov function

$$V(k) = |z_k|_P^2, \ P > 0 \tag{24}$$

and a decay rate $\alpha := 1 - \varepsilon\theta$, where $0 \leq \theta < 1/\varepsilon$. Denote

$$Q_\theta(\varepsilon) := A_{av}^TP + PA_{av} + \theta P + \varepsilon A_{av}^TPA_{av}. \tag{25}$$

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Using (20), we obtain

$$\begin{array}{l} V(k+1)-\alpha V(k)=\varepsilon|z_k|^2_{Q_\theta}+\varepsilon^2\mathcal{Y}^T_{\rho,a}(k)\mathcal{A}_1^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k)\\ +\varepsilon^2\mathcal{Y}^{(1),T}_\rho(k)(A_{av}\mathcal{A})^TP(A_{av}\mathcal{A})\mathcal{Y}^{(1)}_\rho(k)\\ +\varepsilon^2\mathcal{Y}^{(2),T}_\rho(k)(I_2\otimes A_{av})^T\mathcal{A}^TP\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}^{(2)}_\rho(k)\\ +2\varepsilon z_k^T(I+\varepsilon A_{av})^TP(A_{av}\mathcal{A})\mathcal{Y}^{(1)}_\rho(k)\\ -2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}^{(2)}_\rho(k)\\ -2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k)\\ -2\varepsilon^2\mathcal{Y}^{(1),T}_\rho(k)(A_{av}\mathcal{A})^TP\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}^{(2)}_\rho(k)\\ -2\varepsilon^2\mathcal{Y}^{(1),T}_\rho(k)(A_{av}\mathcal{A})^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k)\\ +2\varepsilon^2\mathcal{Y}^{(2),T}_\rho(k)(I_2\otimes A_{av})^T\mathcal{A}^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k). \end{array} \tag{26}$$

Remark 3. *In* (26) *we want to obtain* $V(k+1)-\alpha V(k)=\varepsilon|z_k|^2_{Q_\theta}+O(\varepsilon^2)$. *The dominating term* $\varepsilon|z_k|^2_{Q_\theta}$ *is essential for deriving LMIs that are feasible for small* $\epsilon>0$. *In order to achieve the former relation,* α *is taken in the form* $\alpha=1-\varepsilon\theta$. *Note that system* (4) *is slow and its decay rate is small (see also [18]). Hence, the choice* $\alpha=1-\varepsilon\theta$ *is consistent with this observation.*

Using (19), we present (26) as a quadratic in x_k via:

$$\begin{array}{l} |z_k|^2_{\varepsilon Q_\theta(\varepsilon)}=|x_k|^2_{\varepsilon Q_\theta(\varepsilon)}-2x_k^T\varepsilon Q_\theta(\varepsilon)\mathcal{A}\mathcal{Y}^{(1)}_\rho(k)\\ +\varepsilon\mathcal{Y}^{(1),T}_\rho(k)\mathcal{A}^TQ_\theta(\varepsilon)\mathcal{A}\mathcal{Y}^{(1)}_\rho(k), \end{array} \tag{27}$$

$$\begin{array}{l} 2\varepsilon z_k^T(I+\varepsilon A_{av})^TP(A_{av}\mathcal{A})\mathcal{Y}^{(1)}_\rho(k)=\\ -\varepsilon\mathcal{Y}^{(1),T}_\rho(k)\mathcal{A}^T(I+\varepsilon A_{av})^TP(A_{av}\mathcal{A})\mathcal{Y}^{(1)}_\rho(k)\\ -\varepsilon\mathcal{Y}^{(1),T}_\rho(k)(A_{av}\mathcal{A})P(I+\varepsilon A_{av})\mathcal{A}\mathcal{Y}^{(1)}_\rho(k)\\ +2\varepsilon x_k^T(I+\varepsilon A_{av})^TP(A_{av}\mathcal{A})\mathcal{Y}^{(1)}_\rho(k), \end{array} \tag{28}$$

$$\begin{array}{l} -2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}^{(2)}_\rho(k)=\\ -2\varepsilon x_k^T(I+\varepsilon A_{av})^TP\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}^{(2)}_\rho(k)\\ +2\varepsilon\mathcal{Y}^{(1),T}_\rho(k)\mathcal{A}^T(I+\varepsilon A_{av})^TP\mathcal{A}(I_2\otimes A_{av})\mathcal{Y}^{(2)}_\rho(k), \end{array} \tag{29}$$

$$\begin{array}{l} -2\varepsilon z_k^T(I+\varepsilon A_{av})^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k)=\\ -2\varepsilon x_k^T(I+\varepsilon A_{av})^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k)\\ +2\varepsilon\mathcal{Y}^{(1),T}_\rho(k)\mathcal{A}^T(I+\varepsilon A_{av})^TP\mathcal{A}_1\mathcal{Y}_{\rho,a}(k). \end{array} \tag{30}$$

Let

$$\begin{array}{l} \eta(k)=col\{x_k,\mathcal{Y}^{(1)}_\rho(k),\mathcal{Y}^{(2)}_\rho(k),\mathcal{Y}_{\rho,a}(k)\},\\ W_1=|\Lambda^{(1)}_\rho H_\rho|_1|x_k|^2-\mathcal{Y}^{(1),T}_\rho(k)(\Lambda^{(1)}_\rho\otimes I_n)\mathcal{Y}^{(1)}_\rho(k),\\ W_2=|\Lambda^{(2)}_\rho H_\rho|_1|x_k|^2-\mathcal{Y}^{(2),T}_\rho(k)(\Lambda^{(2)}_\rho\otimes I_n)\mathcal{Y}^{(2)}_\rho(k),\\ W_3=|\Lambda_{\rho,a}H_{\rho,a}|_1|x_k|^2-\mathcal{Y}^T_{\rho,a}(k)(\Lambda_{\rho,a}\otimes I_n)\mathcal{Y}_{\rho,a}(k). \end{array} \tag{31}$$

Then, (23) implies that $W_i\geq 0$ for all $i=1,2,3$. Using (23)-(31) and the S-procedure [5], we arrive at

$$\begin{array}{l} V(k+1)-\alpha V(k)\leq V(k+1)-\alpha V(k)+\varepsilon\sum_{m=1}^3 W_m\\ \leq\varepsilon\eta^T(k)\Phi_\varepsilon\eta(k)\leq 0, \end{array} \tag{32}$$

Provided

$$\Phi_\varepsilon=\begin{bmatrix}\beta_1 & B\\ * & \Psi_\varepsilon\end{bmatrix}<0, \tag{33}$$

where

$$\begin{array}{l} \Psi_\varepsilon=\begin{bmatrix}\phi_1 & \phi_2 & \phi_3\\ * & \phi_4 & \varepsilon(I_2\otimes A_{av})^T\mathcal{A}^TP\mathcal{A}_1\\ * & * & -(\Lambda_{\rho,a}\otimes I_n)+\varepsilon\mathcal{A}_1^TP\mathcal{A}_1\end{bmatrix},\\ B=\begin{bmatrix}\beta_2 & \beta_3 & \beta_4\end{bmatrix},\\ \beta_1=Q_\theta+\sum_{i=1}^2|\Lambda^{(i)}_\rho H_\rho|_1I_n+|\Lambda_{\rho,a}H_{\rho,a}|_1I_n,\\ \beta_2=-Q_\theta\mathcal{A}+(I_n+\varepsilon A_{av})^TPA_{av}\mathcal{A},\\ \beta_3=-(I_n+\varepsilon A_{av})^TP\mathcal{A}(I_2\otimes A_{av}),\\ \beta_4=-(I_n+\varepsilon A_{av})^TP\mathcal{A}_1,\\ \phi_1=-(\Lambda^{(1)}_\rho\otimes I_n)+\mathcal{A}^TQ_\theta(\varepsilon)\mathcal{A}\\ +\varepsilon(A_{av}\mathcal{A})^TP(A_{av}\mathcal{A})-(A_{av}\mathcal{A})^TP(I+\varepsilon A_{av})\mathcal{A}\\ -\mathcal{A}^T(I_n+\varepsilon A_{av})^TP(A_{av}\mathcal{A}),\\ \phi_2=\mathcal{A}^T(I_n+\varepsilon A_{av})^TP\mathcal{A}(I_2\otimes A_{av})\\ -\varepsilon(A_{av}\mathcal{A})^TP\mathcal{A}(I_2\otimes A_{av}),\\ \phi_3=\mathcal{A}^T(I_n+\varepsilon A_{av})^TP\mathcal{A}_1-\varepsilon(A_{av}\mathcal{A})^TP\mathcal{A}_1,\\ \phi_4=-(\Lambda^{(2)}_\rho\otimes I_n)+\varepsilon(I_2\otimes A_{av})^T\mathcal{A}^TP\mathcal{A}(I_2\otimes A_{av}) \end{array} \tag{34}$$

Summarizing, we arrive at:

Theorem 1. *Consider system* (4) *subject to Assumption 1, let* $H_\rho,H_{\rho,a}$ *be defined by* (22). *Given tuning parameters* $\theta>0$ *and* $\varepsilon^*>0$ *subject to* (11) *and* $\theta\varepsilon^*<1$. *Let there exist* $0<P\in\mathbb{R}^{n\times n}$, *and diagonal positive matrices* $\Lambda^{(1)}_\rho,\Lambda^{(2)}_\rho\in\mathbb{R}^{2\times 2}$ *and* $\Lambda_{\rho,a}\in\mathbb{R}^{4\times 4}$ *such that LMI* (33) *with notations* (25) *and* (34) *holds for* $\varepsilon=\varepsilon^*$. *Then, system* (4) *is exponentially stable with a decay rate* $\sqrt{1-\theta\varepsilon}$ *for all* $\varepsilon\in(0,\varepsilon^*]$, *namely, there exists a* $M>0$ *such that for all* $\varepsilon\in(0,\varepsilon^*]$, *the solution of* (4) *initialized by* $x_0\in\mathbb{R}^n$ *satisfies*

$$|x_k|^2\leq M(1-\theta\varepsilon)^k|x_0|^2,\ \forall k\in\mathbb{Z}_+. \tag{35}$$

Moreover, if (11) *and* (33) *hold with* $\varepsilon=\varepsilon^*$ *and* $\theta=0$, *then* (4) *is exponentially stable for all* $\varepsilon\in(0,\varepsilon^*]$. *The inequalities* (11) *and* (33) *are always feasible for small enough* ε *and* θ.

Proof. The fact that given the feasibility of Φ_ε for some θ,ε^* implies its feasibility for all $\varepsilon<\varepsilon^*$, follows from monotonicity of the matrix Φ_ε with respect to $\varepsilon<\varepsilon^*$ (meaning that as the small parameter decreases, the eigenvalues of Φ_ε are non-increasing).

Feasibility of Φ_ε implies that for all

$$V(k+1)-\alpha V(k)\leq 0\Rightarrow V(k+1)\leq\alpha^{k+1}V(0).$$

Since $\lambda_{min}(P)|z_k|^2\leq V(k)\leq\lambda_{max}(P)|z_k|^2$ for all $k\in\mathbb{Z}$, we have

$$|z_{k+1}|^2\leq\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\alpha^{k+1}|z_0|^2. \tag{36}$$

Inequality (36) together with (14) and (15) yield

$$|x_{k+1}|^2\leq\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\delta_1^2(1+\delta_2)^2(1-\varepsilon\theta)^{k+1}|x_0|^2. \tag{37}$$

Thus, we obtain (35) with $M:=\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\delta_1^2(1+\delta_2)^2$. For the LMI (33) feasibility guarantees, we choose $\Lambda^{(1)}_\rho=\Lambda^{(1)}_\rho=\lambda I_2$ and $\lambda_{\rho,a}=\lambda I_4$, where $\lambda>0$. We begin by choosing $\theta=0$ (keep in mind $|H_{\rho,a}|_1=O(\varepsilon^2)$, $|H_{\rho,a}|_1=O(\varepsilon^2)$), since A_{av} is Hurwitz matrix, there is $0<P\in\mathbb{R}^n$ such that $\beta_1<0$ for small enough ε (see (25), Assumption 1). It can be shown that $\Psi_\varepsilon<0$ for large enough λ and small enough ε (the diagonal elements are linear and negative in λ). Next, we apply Schur complement to Φ_ε, whence $\Phi_\varepsilon<0$ iff $\beta_1-\frac{1}{\lambda}B(\lambda^{-1}\Psi_\varepsilon)^{-1}B^T<0$. Note that $-(\lambda^{-1}\Psi_\varepsilon)^{-1}$ is bounded as $\lambda\to\infty$ (converges to the identity matrix), whereas B is independent of λ. Thus, the feasibility of Φ_ε is guaranteed. □

Numerical Example: Consider a suspended pendulum with suspension point that is subject to vertical vibrations of small amplitude and high frequency. The discrete-time version of the linearized model at the upper equilibrium position with a sampling period h is described as (4) with

$$A(k)=h\begin{bmatrix}\cos(kh) & 1\\ 0.04-\cos^2(kh) & -0.2-\cos(kh)\end{bmatrix}. \tag{38}$$

The matrix $A(k)$ given by (38) can be presented as (5) with $a_1(h)=\cos(kh)$, $a_2(k)=\cos(2kh)$ and

$$A_{av}=\begin{bmatrix}0 & h\\ \frac{h}{25}-\frac{h}{2} & -\frac{h}{5}\end{bmatrix},\ A_1=\begin{bmatrix}h & 0\\ 0 & -h\end{bmatrix},A_2=\begin{bmatrix}0 & 0\\ -\frac{h}{2} & 0\end{bmatrix}. \tag{39}$$

The corresponding sampling periods is $h=\frac{\pi}{20}$ and $T=40$. We consider $\theta\in\{0,0.01\}$ and verify the LMI $\Phi_{\varepsilon^*}<0$ of Theorem 1 to obtain the maximal value ε^* which guarantees the exponential stability of system (4) with (38). With some simply calculations, it is easily shown that (11) holds for the obtained ε^* (see Table I). As seen from Table I, the resulting values of ε^* are essentially larger than those obtained in [18].

Method	$\theta = 0$	$\theta = 0.01$
Yang, Zhang & Fridman	$0.71 \cdot 10^{-2}$	$0.47 \cdot 10^{-2}$
Theorem 1	$2.696 \cdot 10^{-2}$	$1.840 \cdot 10^{-2}$

Table I
MAXIMUM ε^* PRESERVING THE LMI FEASIBILITY

Remark 4. *Note that the significant improvement in the maximal value of ε^* is achieved due to the new system presentation* (5) *and the novel transformation* (10) *which yield simpler analysis and reduce the conservatism on the upper bounds of the system parameters.*

III. STABILITY OF THE DISCRETE-TIME SYSTEMS WITH DELAYS

Using the new transformation, which is based on summation of the rapidly varying coefficients only (and does not include the state inside of the summation, as was done in the time-delay approach [18]), we present the first stability conditions for discrete-time systems with delays. These conditions are essentially more efficient for constant than for time-varying delays.

We use the simple Lyapunov functionals and the corresponding analysis following Chapter 6 in [5], which still lead to large delays in the examples because the system under consideration is slow. Less conservative results with essentially more complicated LMIs can be derived by using recent analysis e.g. in [16], [20].

A. Constant Delay

Consider the system

$$x_{k+1} = (I_n + \varepsilon A_0)x_k + \varepsilon A_D(k)x_{k-d}, \;\; k \in \mathbb{Z}_+, \tag{40}$$

where d is a positive integer.

Assumption 2. *Assume that $A_D(k),\ k \in \mathbb{Z}_+$ is of the form:*

$$A_D(k) = A_d + \sum_{i=1}^{N_d} a_i(k)A_i, \tag{41}$$

whereas $\{a_i(k)\}_{i=1}^{N_d}$ are T-periodic with the zero average (i.e. satisfy (6)*). In addition, we assume that $A_0 + A_d$ is a Hurwitz matrix.*

As in Section II, we will derive constructive stability conditions by using two steps - system transformation and Lyapunov analysis.

Step I: System Transformation. We modify the transformation (10) to account for the delay:

$$z_k = x_k - \sum_{m=1}^{N_d} \rho_m(k)A_m x_{k-d}, \;\; k \geq d. \tag{42}$$

We will proceed with the case $N_d = 2$ for the simplicity of the presentation. The general case follows the same arguments. Let

$$A_{av} := A_0 + A_d, \quad \xi_k := x_k - x_{k-d}, \; \forall k \in \mathbb{Z}_+.$$

By employing (6), (9), (40) - (42) we obtain

$$\begin{array}{r} z_{k+1} - z_k = \varepsilon A_{av} z_k - \varepsilon A_d \xi_k \\ +\varepsilon \sum_{m=1}^2 A_{av} A_m \rho_m(k) x_{k-d} \\ -\varepsilon \sum_{m=1}^2 A_m A_0 \rho_m(k+1) x_{k-d} \\ -\varepsilon \sum_{m=1}^2 A_m A_d \rho_m(k+1) x_{k-2d} \\ -\varepsilon \sum_{m=1}^2 \sum_{i=1}^2 A_m A_i \rho_m(k+1) a_i(k-d) x_{k-2d}, \end{array} \tag{43}$$

Since $\rho_j(k) = O(\varepsilon)$, equation (43) has the form

$$z_{k+1} - z_k = \varepsilon A_{av} z_k - \varepsilon A_d \xi_k + O(\varepsilon^2). \tag{44}$$

Denote

$$\begin{array}{c} \mathcal{Y}_{a,d}(k) = col\{a_j(k)x_{k-d}\}_{j=1}^2, \\ \mathcal{Y}_{\rho,d}^{(m)}(k) = col\{\rho_j(k+m-1)x_{k-d}\}_{j=1}^2, \; m = 1,2, \\ \mathcal{Y}_{\rho,d}(k) = col\{\mathcal{Y}_{\rho,d}^{(j)}(k)\}_{j=1}^2, \\ \mathcal{Y}_{\rho,2d}(k) = col\{\rho_j(k+1)x_{k-2d}\}_{j=1}^2, \\ \mathcal{Y}_{\rho,a,d}(k) = col\{\rho_1(k+1)a_1(k-d)x_{k-2d}, \\ \rho_1(k+1)a_2(k-d)x_{k-2d}, \rho_2(k+1)a_1(k-d)x_{k-2d}, \\ \rho_2(k+1)a_2(k-d)x_{k-2d}\}, \end{array} \tag{45}$$

$$\begin{array}{c} \mathcal{A} = [A_1, A_2], \; \mathcal{A}_1 = [A_1A_1, A_1A_2, A_2A_1, A_2A_2], \\ \tilde{\mathcal{A}} = [\mathcal{A}, 0_{2\times 2}, 0_{2\times 2}], \mathcal{A}_{\rho,d} = [A_{av}\mathcal{A}, -\mathcal{A}(I_2 \otimes A_0)]. \end{array} \tag{46}$$

Then, (43) can be presented as

$$\begin{array}{r} z_{k+1} - z_k = \varepsilon A_{av} z_k - \varepsilon A_d \xi_k + \varepsilon \mathcal{A}_{\rho,d}\mathcal{Y}_{\rho,d}(k) \\ -\varepsilon \mathcal{A}(I_2 \otimes A_d)\mathcal{Y}_{\rho,2d}(k) - \varepsilon \mathcal{A}_1 \mathcal{Y}_{\rho,a,d}(k), \end{array} \tag{47}$$

whereas due to (40), (42) we have

$$x_{k+1} - x_k = \varepsilon A_{av} x_k - \varepsilon A_d \xi_{k-d} + \varepsilon \mathcal{A}\mathcal{Y}_{a,d}(k), \tag{48}$$

$$z_k = x_k - \tilde{\mathcal{A}}\mathcal{Y}_{\rho,d}(k). \tag{49}$$

Let

$$H_a = col(h_a^{(1)}, h_a^{(2)}) \tag{50}$$

and H_ρ, $H_{\rho,a}$ be as in (21), where $h_a^{(i)}$, $h_\rho^{(i)}$, $h_{\rho,a}^{(i,j)}$ satisfy,

$$\begin{array}{c} a_i^2(k) \leq h_a^{(i)}, \quad \rho_i^2(k) \leq h_\rho^{(i)}, \; \forall k \in \mathbb{Z}_+, \\ \rho_i^2(k+1)a_j^2(k-d) \leq h_{\rho,a}^{(i,j)}, \; \forall i,j \in \{1,2\}. \end{array} \tag{51}$$

Since $\rho_j(k) = O(\varepsilon)$, we have $|H_\rho|_1 = O(\varepsilon^2)$ and $|H_{\rho,a}|_1 = O(\varepsilon^2)$. Then, for any diagonal positive matrices $\Lambda_a, \Lambda_{\rho,2d} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\rho,a}, \Lambda_{\rho,d} \in \mathbb{R}^{4\times 4}$ the following holds:

$$\begin{array}{c} \mathcal{Y}_{a,d}^T(k)(\Lambda_a \otimes I_n)\mathcal{Y}_{a,d}(k) \leq |\Lambda_a H_a|_1 |x_{k-d}|^2, \\ \mathcal{Y}_{\rho,d}^T(k)(\Lambda_{\rho,d} \otimes I_n)\mathcal{Y}_{\rho,d}(k) \leq |\Lambda_{\rho,d}(I_2 \otimes H_\rho)|_1 |x_{k-d}|^2, \\ \mathcal{Y}_{\rho,2d}^T(k)(\Lambda_{\rho,2d} \otimes I_n)\mathcal{Y}_{\rho,2d}(k) \leq |\Lambda_{\rho,2d} H_\rho|_1 |x_{k-2d}|^2, \\ \mathcal{Y}_{\rho,a,d}^T(k)(\Lambda_{\rho,a} \otimes I_n)\mathcal{Y}_{\rho,a,d}(k) \leq |\Lambda_{\rho,a} H_{\rho,a}|_1 |x_{k-2d}|^2. \end{array} \tag{52}$$

The matrices $\Lambda_a, \Lambda_{\rho,a}$, $\Lambda_{\rho,d}$ and $\Lambda_{\rho,2d}$ will be decision variables in the LMIs derived below (see (66)).

Step II: Lyapunov Analysis. For stability analysis of (47) subject to (49), let

$$V_P(k) = |z_k|_P^2, \;\; P > 0, \tag{53}$$

$$V_{S_m,md}(k) = \textstyle\sum_{i=k-md}^{k-1} \alpha^{k-i-1}|x_i|_{S_m}^2, S_m > 0, \tag{54}$$

$$V_{R,d}(k) = d\textstyle\sum_{i=-d}^{-1}\sum_{s=k+i}^{k-1} \alpha^{k-s-1}|x_{s+1} - x_s|_R^2, R > 0 \tag{55}$$

where $m = 1,2$ and $\alpha := 1 - \varepsilon\theta$ is the desired decay rate with $0 \leq \theta < 1/\varepsilon$. Introduce the Lyapunov function

$$V(k) = V_P(k) + \varepsilon\left(\sum_{m=1}^2 V_{S_m,md}(k) + V_{R,d}(k)\right), \tag{56}$$

Here $V_{S_1,d}$ and $V_{R,d}$ compensate x_{k-d}, whereas $V_{S_2,2d}$ compensates x_{k-2d} in the stability analysis. Then,

$$\begin{array}{r} V_P(k+1) - \alpha V_P(k) = \varepsilon |z_k|_{Q_\theta}^2 + \varepsilon^2 \xi_k^T A_d^T P A_d \xi_k \\ +\varepsilon^2 \mathcal{Y}_{\rho,d}(k)^T \mathcal{A}_{\rho,d}^T P \mathcal{A}_{\rho,d}\mathcal{Y}_{\rho,d}(k) \\ +\varepsilon^2 \mathcal{Y}_{\rho,a,d}(k)^T \mathcal{A}_1^T P \mathcal{A}_1 \mathcal{Y}_{\rho,a,d}(k) \\ +\varepsilon^2 \mathcal{Y}_{\rho,2d}(k)^T [\mathcal{A}(I_2 \otimes A_d)]^T P [\mathcal{A}(I_2 \otimes A_d)]\mathcal{Y}_{\rho,2d}(k) \\ -2\varepsilon z_k^T (I + \varepsilon A_{av})^T P A_d \xi_k \\ +2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}_{\rho,d}\mathcal{Y}_{\rho,d}(k) \\ -2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Y}_{\rho,2d}(k) \\ -2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}_1 \mathcal{Y}_{\rho,a,d}(k) \\ -2\varepsilon^2 \xi_k^T A_d^T P \mathcal{A}_{\rho,d}\mathcal{Y}_{\rho,d}(k) + 2\varepsilon^2 \xi_k^T A_d^T P \mathcal{A}_1 \mathcal{Y}_{\rho,a,d}(k) \\ +2\varepsilon^2 \xi_k^T A_d^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Y}_{\rho,2d}(k) \end{array} \tag{57}$$

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$$-2\varepsilon^2\mathcal{Y}_{\rho,d}(k)^T\mathcal{A}_{\rho,d}^T P\mathcal{A}(I_2\otimes A_d)\mathcal{Y}_{\rho,2d}(k) -2\varepsilon^2\mathcal{Y}_{\rho,d}(k)^T\mathcal{A}_{\rho,d}^T P\mathcal{A}_1\mathcal{Y}_{\rho,a,d}(k) +2\varepsilon^2\mathcal{Y}_{\rho,2d}(k)^T[\mathcal{A}(I_2\otimes A_d)]^T P\mathcal{A}_1\mathcal{Y}_{\rho,a,d}(k).$$

Here Q_θ is defined in (25). Substitute $z_k = x_k - \tilde{\mathcal{A}}\mathcal{Y}_{\rho,d}(k)$, we have

$$|z_k|^2_{Q_\theta} = |x_k|^2_{Q_\theta} + |\mathcal{Y}_{\rho,d}(k)|^2_{\tilde{\mathcal{A}}^T Q_\theta \tilde{\mathcal{A}}} - 2x_k^T Q_\theta \tilde{\mathcal{A}}\mathcal{Y}_{\rho,d}(k), \quad (58)$$

$$\begin{aligned} &-2\varepsilon z_k^T(I_n+\varepsilon A_{av})^T P[A_d\xi_k - \mathcal{A}_{\rho,d}\mathcal{Y}_{\rho,d}(k) \\ &\quad +\mathcal{A}(I_2\otimes A_d)\mathcal{Y}_{\rho,2d}(k) + \mathcal{A}_1\mathcal{Y}_{\rho,a,d}(k)] = \\ &-2\varepsilon[x_k - \tilde{\mathcal{A}}\mathcal{Y}_{\rho,d}(k)]^T(I+\varepsilon A_{av})^T P\cdot[A_d\xi_k \\ &\quad -\mathcal{A}_{\rho,d}\mathcal{Y}_{\rho,d}(k) + \mathcal{A}(I_2\otimes A_d)\mathcal{Y}_{\rho,2d}(k) + \mathcal{A}_1\mathcal{Y}_{\rho,a,d}(k)]. \end{aligned} \quad (59)$$

Note that

$$V_{S_1,d}(k+1) - \alpha V_{S_1,d}(k) = (1-\alpha^d)|x_k|^2_{S_1} -\alpha^d|\xi_k|^2_{S_1} + 2\alpha^d x_k^T S_1\xi_k, \quad (60)$$

$$V_{S_2,2d}(k+1) - \alpha V_{S_2,2d}(k) = |x_k|^2_{S_2} - \alpha^{2d}|x_{k-2d}|^2_{S_2}. \quad (61)$$

Let

$$\begin{aligned} \mathcal{L} &= [A_{av}, -A_d, 0_{n\times 4n}, 0_{n\times 2n}, \mathcal{A}, 0_{n\times 4n}], \\ \eta_k &= col\{x_k, \xi_k, \mathcal{Y}_{\rho,d}(k), \mathcal{Y}_{\rho,2d}(k), \mathcal{Y}_{a,d}(k), \mathcal{Y}_{\rho,a,d}(k)\}. \end{aligned} \quad (62)$$

By Jensen's inequality (1), we obtain

$$V_{R,d}(k+1) - \alpha V_{R,d}(k) \le \varepsilon^2 d^2 \eta_{k,d}^T \mathcal{L}^T R\mathcal{L}\eta_{k,d} - \alpha^d \xi_k^T R\xi_k. \quad (63)$$

Denote

$$\begin{aligned} \Pi^{(1)} &= diag\{0, \Lambda_{\rho,d}, \Lambda_{\rho,2d}, \Lambda_a, \Lambda_{\rho,a}\}\otimes I_n, \\ \Pi &= diag\{0_n, \Pi^{(1)}\},\ \lambda_d = |\Lambda_a H_a|_1 + |\Lambda_{\rho,d}(I_2\otimes H_\rho)|_1, \\ \lambda_{2d} &= |\Lambda_{\rho,2d}H_\rho|_1 + |\Lambda_{\rho,a}H_{\rho,a}|_1, \\ W &= -\varepsilon\eta_k^T\Pi\eta_k + \varepsilon\lambda_{2d}|x_{k-2d}|^2 + \varepsilon\lambda_d|x_k-\xi_k|^2. \end{aligned} \quad (64)$$

Inequality (52) implies that $W \ge 0$. Using (52)-(64) and the S-procedure, we arrive at

$$\begin{aligned} V(k+1) - \alpha V(k) &\le V(k+1) - \alpha V(k) + W \\ &\le \varepsilon\eta_k^T(\Theta_{\varepsilon,d} + \varepsilon^2 d^2\mathcal{L}^T R\mathcal{L})\eta_k \\ &\quad +\varepsilon x_{k-2d}^T(-\alpha^{2d}S_2 + \lambda_{2d}I_n)x_{k-2d} \le 0, \end{aligned} \quad (65)$$

provided

$$\Theta_{\varepsilon,d} + \varepsilon^2 d^2\mathcal{L}^T R\mathcal{L} < 0, \quad -\alpha^{2d}S_2 + \lambda_{2d}I_n < 0, \quad (66)$$

where

$$\begin{aligned} &\Theta_{\varepsilon,d} = \begin{bmatrix}\beta_1 & B \\ * & \Psi_{\varepsilon,d}\end{bmatrix}, B = \begin{bmatrix}\beta_2 & \beta_3 & \beta_4 & 0_{n\times 2n} & \beta_5\end{bmatrix}, \\ &\Psi_{\varepsilon,d} = -\Pi^{(1)} + \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & 0_{n\times 2n} & \varepsilon\mathcal{A}_d^T P\mathcal{A}_1 \\ * & \omega_4 & \omega_5 & 0_{4n\times 2n} & \omega_6 \\ * & * & \omega_7 & 0_{2n\times 2n} & \omega_8 \\ * & * & * & 0_{2n\times 2n} & 0_{2n\times 4n} \\ * & * & * & * & \varepsilon\mathcal{A}_1^T P\mathcal{A}_1 \end{bmatrix}, \\ &\omega_1 = \varepsilon A_d^T PA_d - \alpha^d(S_1+R) + \lambda_d I_n, \quad \alpha = 1-\theta\varepsilon, \\ &\omega_2 = A_d^T P(I_n+\varepsilon A_{av})\tilde{\mathcal{A}} - \varepsilon A_d^T P\mathcal{A}_{\rho,d}, \\ &\omega_3 = \varepsilon A_d^T P\mathcal{A}(I_2\otimes A_d), \\ &\omega_4 = \varepsilon\mathcal{A}_{\rho,d}^T P\mathcal{A}_{\rho,d} - \tilde{\mathcal{A}}^T(I_n+\varepsilon A_{av})^T P\mathcal{A}_{\rho,d} \\ &\qquad +\tilde{\mathcal{A}}^T Q_\theta\tilde{\mathcal{A}} - \mathcal{A}_{\rho,d}^T P(I_n+\varepsilon A_{av})\tilde{\mathcal{A}}, \\ &\omega_5 = -\varepsilon\mathcal{A}_{\rho,d}^T P\mathcal{A}(I_2\otimes A_d) \\ &\qquad +\tilde{\mathcal{A}}^T(I_n+\varepsilon A_{av})^T P\mathcal{A}(I_2\otimes A_d), \\ &\omega_6 = -\varepsilon\mathcal{A}_{\rho,d}^T P\mathcal{A}_1 + \tilde{\mathcal{A}}^T(I_n+\varepsilon A_{av})^T P\mathcal{A}_1, \\ &\omega_7 = \varepsilon[\mathcal{A}(I_2\otimes A_d)]^T P[\mathcal{A}(I_2\otimes A_d)], \\ &\omega_8 = \varepsilon[\mathcal{A}(I_2\otimes A_d)]^T P\mathcal{A}_1, \\ &\beta_1 = Q_\theta + (1-\alpha^d)S_1 + S_2 + \lambda_d I_n, \\ &\beta_2 = \alpha^d S_1 - (I_n+\varepsilon A_{av})^T PA_d - \lambda_d I_n, \\ &\beta_3 = -Q_\theta\tilde{\mathcal{A}} + (I_n+\varepsilon A_{av})^T P\mathcal{A}_{\rho,d}, \\ &\beta_4 = -(I_n+\varepsilon A_{av})^T P\mathcal{A}(I_2\otimes A_d), \\ &\beta_5 = -(I_n+\varepsilon A_{av})^T P\mathcal{A}_1, \end{aligned} \quad (67)$$

Summarizing, we arrive at:

Theorem 2. *Consider system* (40) *subject to Assumption 2, let $H_a, H_\rho, H_{\rho,a}$, be defined by* (51). *Given positive tuning parameters θ, d^*, and ε^* subject to $\delta_2 < \alpha^{d^*}$ and $\theta\varepsilon^* < 1$. Let there exist $0 < P, S_1, S_2, R \in \mathbb{R}^{n\times n}$, and diagonal positive matrices $\Lambda_a, \Lambda_\rho^{(1)}, \Lambda_\rho^{(2)}, \Lambda_{\rho,2d} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\rho,a}\in\mathbb{R}^{4\times 4}$ such that LMIs* (66) *with notation* (25) *and* (67) *holds with $\varepsilon = \varepsilon^*$ and $d = d^*$. Then system* (40) *is exponentially stable with a decay rate $\sqrt{1-\theta\varepsilon}$ for all $\varepsilon\in(0,\varepsilon^*]$ and $0\le d\le d^*$. Namely, there exists $M>0$ such that for all $\varepsilon\in(0,\varepsilon^*]$ and $0\le d\le d^*$, the solution of* (40) *initialized at $\{x_j\}_{j=-d}^0$ satisfies*

$$|x_k|^2 \le M\,|x|^2_{[-d,0]}\,(1-\theta\varepsilon)^k,\ \forall k\in\mathbb{Z}_+. \quad (68)$$

Moreover, if LMIs (66) *and $\delta_2 < \alpha^d$ hold with $\varepsilon=\varepsilon^*$, $d=d^*$ and $\theta=0$, then* (40) *is exponentially stable for all $\varepsilon\in(0,\varepsilon^*]$ and $0\le d\le d^*$. Also, given any d, LMI* (66) *and inequality $\delta_2<\alpha^d$ are always feasible for small enough ε and θ.*

Proof. The fact that feasibility of (66) and $\delta_2 < \alpha^d$ with ϵ^*, d^* implies feasibility for all $\varepsilon<\varepsilon^*$, $d<d^*$, and $\delta_2<\alpha^d$ with respect to $\varepsilon<\varepsilon^*$, $d<d^*$.

Feasibility of (66), and $\delta_2<\alpha^d$ implies that for all $d\le k\in\mathbb{Z}_+$,

$$\begin{aligned} &V(k+1) - \alpha V(k) \le 0 \Rightarrow V(k+1) \le \alpha^{k+1-d}V(d), \\ &V(d) = |z_d|^2_P + \textstyle\sum_{m=1}^2\sum_{i=d-m\cdot d}^{d-1}\alpha^{d-i-1}|x_i|^2_{S_m} \\ &\quad +d\textstyle\sum_{i=-d}^{-1}\sum_{s=d+i}^{d-1}\alpha^{d-s-1}|x_{s+1}-x_s|^2_R. \end{aligned} \quad (69)$$

Also, $V(k)\ge\sigma_{min}(P)|z_k|^2$, for any $d\le k\in\mathbb{Z}_+$. Thus, there exists some $M_1>0$ such that

$$|z_k|^2 \le M_1\,|x|^2_{[-d,0]}\,\alpha^{k-d},\ d\le k\in\mathbb{Z}_+. \quad (70)$$

To conclude the same for the solution x_k of the system (40), for any $j\in\mathbb{Z}_+$, we denote $X_j = |x|^2_{[jd,(j+1)d]}$. From (11), (42) and (70), we find that

$$X_{j+1} \le M_2\alpha^{jd} + \delta_2 X_j, j\in\mathbb{Z}_+,$$

where $M_2 = M_1\,|x|^2_{[-d,0]}$. Set $Y_1 = X_1$ and consider the linear difference equation

$$Y_{j+1} = M_2\alpha^{jd} + \delta_2 Y_j,\ j\in\mathbb{Z}_+. \quad (71)$$

By induction, we obtain $X_j\le Y_j$ for all $j\in\mathbb{Z}_+$. Moreover, the solution of (71) is given by $Y_j = \mu_d\alpha^{(j-1)d} + \delta_2^{j-1}(X_1-\mu_d),\ j\in\mathbb{Z}_+$, where $\mu_d = \frac{M_2\alpha}{\alpha-\delta_2}$. Note that Y_k is decreasing. Let $k\in\mathbb{Z}_+$ such that $k\in\mathbf{I}[jd,(j+1)d]$. Then

$$\begin{aligned} |x_k|^2 &\le X_j \le \delta_2^{j-1}(X_1-\mu_d) + \mu_d\alpha^{jd-d} \\ &\le \delta_2^{\frac{k-d}{d}}\frac{(X_1-\mu_d)}{\delta_2} + \mu_d\alpha^{-d}\alpha^{k-d} \\ &\le \left(\frac{(X_1-\mu_d)}{\delta_2} + \mu_d\alpha^{-d}\right)\alpha^{k-d},\ \forall k\ge d_M \end{aligned}$$

where the last inequality follows from $\delta_2<\alpha^d$. In addition, since x_k is bounded on the interval $[0,d]$, inequality (68) holds for all $k\in\mathbb{Z}_+$.

For the LMI feasibility guarantees for given d, we choose $\Lambda_a = \Lambda_{\rho,2d} = \lambda_1 I_2$, $\Lambda_{\rho,a} = \Lambda_{\rho,d} = \lambda_1 I_4$, $R=\lambda_1 I_n$, $S_1=\lambda_d I_n$, where $\lambda_1>\lambda_d$, $S_2=\lambda_2 I_n$ where $\lambda_2 = 2\lambda_{2d}$ (keep in mind that $\lambda_{2d} = O(\varepsilon^2)$). For $\theta=0$ (so $\alpha=1$), the inequality $-\alpha^{2d}S_2+\lambda_{2d}I_n<0$ holds, and also $\Theta_{\varepsilon,d}$ is independent of d whereas $\varepsilon^2d^2\mathcal{L}^TR\mathcal{L}$ and $\delta_2<\alpha^d$ (since $\delta_2=O(\varepsilon)$) are small for enough small ε. So, it is sufficient to prove that $\Theta_{\varepsilon,d}<0$. Since A_{av} is Hurwitz matrix, there is a $0<P\in\mathbb{R}^n$ such that $\beta_1<0$ for small enough ε. It is easily seen that $\Psi_{\varepsilon,d}<0$ for large enough λ_1 and small enough $\varepsilon>0$. Next, we apply Schur complement with respect to $\Theta_{\varepsilon,d}$, whence $\Theta_{\varepsilon,d}<0$ iff $\beta_1 - \frac{1}{\lambda_1}B(\lambda_1^{-1}\Psi_{\varepsilon,d})^{-1}B^T < 0$. Note that $-(\lambda_1^{-1}\Psi_{\varepsilon,d})^{-1}$ is bounded as $\lambda_1\to\infty$ (converges to the identity matrix), whereas B and β_1 are independent of λ_1 implying the feasibility of $\Theta_{\varepsilon,d}$ □

B. Time-Varying Delays

Consider the system

$$x_{k+1} = (I_n + \varepsilon A_0)x_k + \varepsilon A_D(k)x_{k-d_k}, \ \ k \in \mathbb{Z}_+, \quad (72)$$

where $d_k \le d_M$, $\forall k \in \mathbb{Z}_+$ and d_M is an integer. Assume that Assumption (2) holds with $N_d = 2$ (for simplicity of the presentation).

Differently from the constant delay, for the time-varying delay we use the same transformation (10) as for the non-delayed case (this follows the analysis in the continuous-time delayed case [9]).

Remark 5. *Note that one can choose a different delayed transformation (for example the one given by* (42)*, where* d *is changed by* d_k*). This may require further constraints on delay variation and will essentially complicate the analysis and the resulting LMIs.*

We will derive constructive stability conditions by using two steps - system transformation and Lyapunov analysis.

Step I: System Transformation. Denote

$$\xi_k = x_{k-d_k} - x_k, \ \nu_k = x_{k-d_M} - x_{k-d_k}.$$

Employing (10) and (72) we obtain the following expression

$$\begin{array}{c} z_{k+1} - z_k = \varepsilon A_{av} z_k + \varepsilon A_d \xi_k + \varepsilon \sum_{m=1}^{2} a_m(k) A_m \xi_k \\ -\varepsilon \sum_{m=1}^{2} \rho_m(k+1) A_m A_{av} x_k \\ -\varepsilon \sum_{m=1}^{2} \rho_m(k+1) A_m A_d \xi_k \\ +\varepsilon \sum_{m=1}^{2} \rho_m(k) A_{av} A_m x_k \\ -\varepsilon \sum_{m=1}^{2}\sum_{i=1}^{2} \rho_m(k+1) a_i(k) A_m A_i x_k \\ -\varepsilon \sum_{m=1}^{2}\sum_{i=1}^{2} \rho_m(k+1) a_i(k) A_m A_i \xi_k. \end{array} \quad (73)$$

Since $\rho_j(k) = O(\varepsilon)$, equation (73) has the form

$$z_{k+1} - z_k = \varepsilon A_{av} z_k + \varepsilon A_d \xi_k + \varepsilon \sum_{m=1}^{2} a_m(k) A_m \xi_k + O(\varepsilon^2). \quad (74)$$

Differently from the constant case (see equation (44)), for time-varying delay case, there are additional terms $\varepsilon \sum_{m=1}^{2} a_m(k) A_m \xi_k$ of order $O(\varepsilon)$ (see equation (74)). Let $\mathcal{Y}_\rho^{(1)}(k)$, $\mathcal{Y}_\rho^{(2)}(k)$ and $\mathcal{Y}_{\rho,a}(k)$ be as in (17), (18). Denote

$$\begin{array}{l} \mathcal{Y}_a(k) = col\{a_j(k)x_k\}_{j=1}^2, \mathcal{Z}_a(k) = col\{a_j(k)\xi_k\}_{j=1}^2, \\ \mathcal{Y}_\rho(k) = col\{\mathcal{Y}_\rho^{(j)}(k)\}_{j=1}^2, \mathcal{Z}_\rho(k) = col\{\rho_j(k+1)\xi_k\}_{j=1}^2, \\ \mathcal{Z}_{\rho,a}(k) = col\{\rho_1(k+1)a_1(k)\xi_k, \rho_1(k+1)a_2(k)\xi_k, \\ \qquad \rho_2(k+1)a_1(k)\xi_k, \rho_2(k+1)a_2(k)\xi_k\}. \end{array}$$

Then, equation (72) can be presented as

$$\begin{array}{l} z_{k+1} - z_k = \varepsilon A_{av} z_k + \varepsilon \mathcal{A}_\rho \mathcal{Y}_\rho(k) - \varepsilon \mathcal{A}_1 \mathcal{Y}_{\rho,a}(k) + \varepsilon A_d \xi_k \\ \quad + \varepsilon \mathcal{A}\mathcal{Z}_a(k) - \varepsilon \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) - \varepsilon \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k), \end{array} \quad (75)$$

whereas due to (10) and (73), we obtain

$$z_k = x_k - \tilde{\mathcal{A}}\mathcal{Y}_\rho(k), \quad (76)$$

$$x_{k+1} - x_k = \varepsilon A_{av} x_k + \varepsilon \mathcal{A}\mathcal{Y}_a(k) + \varepsilon A_d \xi_k + \varepsilon \mathcal{A}\mathcal{Z}_a(k), \quad (77)$$

where $\mathcal{A}$, $\mathcal{A}_1$, $\tilde{\mathcal{A}}$, $\mathcal{A}_{\rho,d}$ as defined in (17), (18), (46). Let H_a, H_ρ, $H_{\rho,a}$ be as in (21), (50), where $h_a^{(i)}$ and $h_\rho^{(i)}, h_{\rho,a}^{(i,j)}$ satisfy the inequalities (22) and (51) respectively. Since $\rho_j(k) = O(\varepsilon)$, we have $|H_\rho|_1 = O(\varepsilon^2)$ and $|H_{\rho,a}|_1 = O(\varepsilon^2)$. Then, for any diagonal positive matrices $\Lambda_{\mathcal{Y}_a}, \Lambda_{\mathcal{Z}_a}, \Lambda_{\mathcal{Z}_\rho} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\mathcal{Y}_\rho}, \Lambda_{\mathcal{Y}_{\rho,a}}, \Lambda_{\mathcal{Z}_{\rho,a}} \in \mathbb{R}^{4\times 4}$ the following holds:

$$\begin{array}{l} \mathcal{Y}_a(k)^T(\Lambda_{\mathcal{Y}_a} \otimes I_n)\mathcal{Y}_a(k) \le |\Lambda_{\mathcal{Y}_a} H_a|_1 |x_k|^2, \\ \mathcal{Z}_a(k)^T(\Lambda_{\mathcal{Z}_a} \otimes I_n)\mathcal{Z}_a(k) \le |\Lambda_{\mathcal{Z}_a} H_a|_1 |\xi_k|^2, \\ \mathcal{Y}_\rho(k)^T(\Lambda_{\mathcal{Y}_\rho} \otimes I_n)\mathcal{Y}_\rho(k) \le |\Lambda_{\mathcal{Y}_\rho}(I_2 \otimes H_\rho)|_1 |x_k|^2, \\ \mathcal{Z}_\rho(k)^T(\Lambda_{\mathcal{Z}_\rho} \otimes I_n)\mathcal{Z}_\rho(k) \le |\Lambda_{\mathcal{Z}_\rho} H_\rho|_1 |\xi_k|^2, \\ \mathcal{Y}_{\rho,a}(k)^T(\Lambda_{\mathcal{Y}_{\rho,a}} \otimes I_n)\mathcal{Y}_{\rho,a}(k) \le |\Lambda_{\mathcal{Y}_{\rho,a}} H_{\rho,a}|_1 |x_k|^2, \\ \mathcal{Z}_{\rho,a}(k)^T(\Lambda_{\mathcal{Z}_{\rho,a}} \otimes I_n)\mathcal{Z}_{\rho,a}(k) \le |\Lambda_{\mathcal{Z}_{\rho,a}} H_{\rho,a}|_1 |\xi_k|^2. \end{array}$$

The matrices $\Lambda_{\mathcal{Y}_a}, \Lambda_{\mathcal{Z}_a}, \Lambda_{\mathcal{Z}_\rho} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\mathcal{Y}_\rho}, \Lambda_{\mathcal{Y}_{\rho,a}}, \Lambda_{\mathcal{Z}_{\rho,a}} \in \mathbb{R}^{4\times 4}$ will be decision variables in the LMIs below (see (87), (88)).

Step II: Lyapunov Analysis. For stability analysis of (75) subject to (76), choose the LK functional

$$V(k) = V_P(k) + \varepsilon\left(V_{S,d_M}(k) + V_{R,d_M}(k)\right), \forall k \ge d_M, \quad (78)$$

where $V_P(k)$, $V_{S,d_M}(k)$ and $V_{R,d_M}(k)$ are as defined in (53), (63). Introduce the decay rate $\alpha := 1 - \varepsilon\theta$, where $0 \le \theta < 1/\varepsilon$. Then,

$$\begin{array}{c} V_P(k+1) - \alpha V_P(k) = \varepsilon |z_k|^2_{Q_\theta} + \varepsilon^2 \mathcal{Y}_\rho(k)^T \mathcal{A}_\rho^T P \mathcal{A}_\rho \mathcal{Y}_\rho(k) \\ +\varepsilon^2 \mathcal{Y}_{\rho,a}(k)^T \mathcal{A}_1^T P \mathcal{A}_1 \mathcal{Y}_{\rho,a}(k) + \varepsilon^2 \xi_k^T A_d^T P A_d \xi_k \\ +\varepsilon^2 \mathcal{Z}_\rho(k)^T [\mathcal{A}(I_2 \otimes A_d)]^T P [\mathcal{A}(I_2 \otimes A_d)] \mathcal{Z}_\rho(k) \\ +\varepsilon^2 \mathcal{Z}_a(k)^T \mathcal{A}^T P \mathcal{A}\mathcal{Z}_a(k) + \varepsilon^2 \mathcal{Z}_{\rho,a}(k)^T \mathcal{A}_1^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k) \\ +2\varepsilon z_k^T (I + \varepsilon A_{av})^T P A_d \xi_k - 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}_1 \mathcal{Y}_{\rho,a}(k) \\ +2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}_\rho \mathcal{Y}_\rho + 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}\mathcal{Z}_a(k) \\ -2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) \\ -2\varepsilon z_k^T (I + \varepsilon A_{av})^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k) + 2\varepsilon^2 \mathcal{Y}_\rho(k)^T \mathcal{A}_\rho^T P A_d \xi_k \\ -2\varepsilon^2 \mathcal{Y}_\rho(k)^T \mathcal{A}_\rho^T P \mathcal{A}_1 \mathcal{Y}_{\rho,a}(k) + 2\varepsilon^2 \mathcal{Y}_\rho(k)^T \mathcal{A}_\rho^T P \mathcal{A}\mathcal{Z}_a(k) \\ -2\varepsilon^2 \mathcal{Y}_\rho(k)^T \mathcal{A}_\rho^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) \\ -2\varepsilon^2 \mathcal{Y}_\rho(k)^T \mathcal{A}_\rho^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k) - 2\varepsilon^2 \mathcal{Y}_{\rho,a}(k)^T \mathcal{A}_1^T P \mathcal{A}\mathcal{Z}_a(k) \\ +2\varepsilon^2 \mathcal{Y}_{\rho,a}(k)^T \mathcal{A}_1^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) \\ +2\varepsilon^2 \mathcal{Y}_{\rho,a}(k)^T \mathcal{A}_1^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k) + 2\varepsilon^2 \xi_k^T A_d^T P \mathcal{A}\mathcal{Z}_a(k) \\ -2\varepsilon^2 \xi_k^T A_d^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) - 2\varepsilon^2 \xi_k^T A_d^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k) \\ -2\varepsilon^2 \mathcal{Z}_a(k)^T \mathcal{A}^T P \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) \\ -2\varepsilon^2 \mathcal{Z}_a(k)^T \mathcal{A}^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k) - 2\varepsilon^2 \mathcal{Y}_{\rho,a}(k)^T \mathcal{A}_1^T P A_d \xi_k \\ +2\varepsilon^2 \mathcal{Z}_\rho(k)^T [\mathcal{A}(I_2 \otimes A_d)]^T P \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k), \end{array} \quad (79)$$

where Q_θ is given by (25). Substitute $z_k = x_k - \tilde{\mathcal{A}}\mathcal{Y}_\rho(k)$, we get

$$|z_k|^2_{Q_\theta} = |x_k|^2_{Q_\theta} + |\mathcal{Y}_\rho(k)|^2_{\tilde{\mathcal{A}}^T Q_\theta \tilde{\mathcal{A}}} - 2x_k^T Q_\theta \tilde{\mathcal{A}}\mathcal{Y}_\rho(k), \quad (80)$$

$$\begin{array}{l} 2\varepsilon z_k^T (I + \varepsilon A_{av})^T P[\mathcal{A}_\rho \mathcal{Y}_\rho - \mathcal{A}_1 \mathcal{Y}_{\rho,a}(k) + A_d \xi_k + \mathcal{A}\mathcal{Z}_a(k) \\ \qquad -\mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) - \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k)] = \\ 2\varepsilon \left(x_k - \tilde{\mathcal{A}}\mathcal{Y}_\rho(k)\right)^T (I + \varepsilon A_{av})^T P[\mathcal{A}_\rho \mathcal{Y}_\rho - \mathcal{A}_1 \mathcal{Y}_{\rho,a}(k) + \\ \qquad A_d \xi_k + \mathcal{A}\mathcal{Z}_a(k) - \mathcal{A}(I_2 \otimes A_d)\mathcal{Z}_\rho(k) - \mathcal{A}_1 \mathcal{Z}_{\rho,a}(k)]. \end{array} \quad (81)$$

We find further

$$\begin{array}{l} V_{S,d_M}(k+1) - \alpha V_{S,d_M}(k) = (1 - \alpha^{d_M})|x_k|^2_S - \alpha^{d_M}|\xi_k|^2_S \\ -\alpha^{d_M}|\nu_k|^2_S - 2\alpha^{d_M} x_k^T S \xi_k - 2\alpha^{d_M} x_k^T S \nu_k - 2\alpha^{d_M} \xi_k^T S \nu_k. \end{array} \quad (82)$$

Let

$$\begin{array}{l} \mathcal{L} = [A_{av}, A_d, 0_n, 0_{n\times 4n}, \mathcal{A}, 0_{n\times 4n}, 0_{n\times 2n}, \mathcal{A}, 0_{n\times 4n}], \\ \eta_k = col\{x_k, \xi_k, \nu_k, \mathcal{Y}_\rho(k), \mathcal{Y}_a(k), \mathcal{Y}_{\rho,a}(k), \\ \qquad \mathcal{Z}_\rho(k), \mathcal{Z}_a(k), \mathcal{Z}_{\rho,a}(k)\}, \end{array} \quad (83)$$

By Jensen's inequality (1) and Park's inequality (3), we obtain

$$\begin{array}{l} V_{R,d_M}(k+1) - \alpha V_{R,d_M}(k) \le \varepsilon^2 d_M^2 \eta_k^T \mathcal{L}^T R \mathcal{L} \eta_k \\ \qquad -\alpha^{d_M} \begin{bmatrix} \xi_k \\ \nu_k \end{bmatrix}^T \begin{bmatrix} R & G \\ * & R \end{bmatrix} \begin{bmatrix} \xi_k \\ \nu_k \end{bmatrix}, \end{array} \quad (84)$$

where $0 < R$ and $G \in \mathbb{R}^{n\times n}$ satisfy LMI (2). Denote

$$\begin{array}{c} \lambda_0^{(1)} = |\Lambda_{\mathcal{Y}_\rho} H_\rho|_1 + |\Lambda_{\mathcal{Y}_a} H_a|_1 + |\Lambda_{\mathcal{Y}_{\rho,a}} H_{\rho,a}|_1, \\ \lambda_0^{(2)} = |\Lambda_{\mathcal{Z}_\rho} H_\rho|_1 + |\Lambda_{\mathcal{Z}_a} H_a|_1 + |\Lambda_{\mathcal{Z}_{\rho,a}} H_{\rho,a}|_1, \\ \Sigma_0^{(1)} = \lambda_0^{(1)} I_n, \Sigma_0^{(2)} = \lambda_0^{(2)} I_n, \\ \Pi^{(1)} = diag\{0, 0, \Lambda_{\mathcal{Y}_\rho}, \Lambda_{\mathcal{Y}_a}, \Lambda_{\mathcal{Y}_{\rho,a}}, \Lambda_{\mathcal{Z}_\rho}, \Lambda_{\mathcal{Z}_a}, \Lambda_{\mathcal{Z}_{\rho,a}}\} \otimes I_n, \\ \Sigma_0 = diag\{\Sigma_0^{(1)}, \Sigma_0^{(2)}, 0_n, 0_{4n}, 0_{2n}, 0_{4n}, 0_{2n}, 0_{2n}, 0_{4n}\}, \\ \Sigma_1 = diag\{0_n, \Pi^{(1)}\}, \ W := \varepsilon \eta_k^T (\Sigma_0 - \Sigma_1) \eta_k. \end{array} \quad (85)$$

Then, (78) implies that $W \ge 0$. Using (78)-(85) and the S-procedure, we arrive at

$$\begin{array}{l} V(k+1) - \alpha V(k) \le V(k+1) - \alpha V(k) + W \\ \qquad \le \varepsilon \eta_k^T (\Theta_{\varepsilon,d_M} + \varepsilon^2 d_M^2 \mathcal{L}^T R \mathcal{L}) \eta_k \le 0, \end{array} \quad (86)$$

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provided (2) and the following inequality hold:

$$\Theta_{\varepsilon,d_M}+\varepsilon^2 d_M^2\mathcal{L}^T R\mathcal{L}<0, \tag{87}$$

where

$$\begin{aligned}
&\Theta_{\varepsilon,d_M}=\begin{bmatrix}\beta_1 & B\\ * & \Psi_{\varepsilon,d_M}\end{bmatrix},\\
&B=[\beta_2,\beta_3,\beta_4,0_{n\times 2n},\beta_6,\beta_7,\beta_8,\beta_9],\\
&\beta_1=Q_\theta+(1-\alpha^{d_M})S+\Sigma_0^{(1)},\\
&\quad\beta_2=-\alpha^{d_M}S+(I+\varepsilon A_{av})^T P A_d,\\
&\beta_3=-\alpha^{d_M}S,\ \beta_4=-Q_\theta\tilde{\mathcal{A}}+(I+\varepsilon A_{av})^T P\mathcal{A}_\rho,\\
&\quad\beta_6=-(I+\varepsilon A_{av})^T P\mathcal{A}_1,\\
&\quad\beta_7=-(I+\varepsilon A_{av})^T P\mathcal{A}(I_2\otimes A_d),\\
&\beta_8=(I+\varepsilon A_{av})^T P\mathcal{A},\ \beta_9=-(I+\varepsilon A_{av})^T P\mathcal{A}_1,\\
&\Psi_{\varepsilon,d_M}=-\Pi^{(1)}+\begin{bmatrix}\Psi^{(1)}_{\varepsilon,d_M} & \Psi^{(2)}_{\varepsilon,d_M} & \Psi^{(3)}_{\varepsilon,d_M}\\ * & \Psi^{(4)}_{\varepsilon,d_M} & \Psi^{(5)}_{\varepsilon,d_M}\\ * & * & \Psi^{(6)}_{\varepsilon,d_M}\end{bmatrix},\\
&\Psi^{(1)}_{\varepsilon,d_M}=\begin{bmatrix}\omega_1 & -\alpha^{d_M}(S+G)\\ * & -\alpha^{d_M}(S+R)\end{bmatrix},\\
&\Psi^{(2)}_{\varepsilon,d_M}=\begin{bmatrix}\omega_2 & 0_{n\times 2n} & -\varepsilon A_d^T P\mathcal{A}_1\\ 0_{n\times 4n} & 0_{n\times 2n} & 0_{n\times 4n}\end{bmatrix},\\
&\Psi^{(3)}_{\varepsilon,d_M}=\begin{bmatrix}-\varepsilon A_d^T P\mathcal{A}(I_2\otimes A_d) & \varepsilon A_d^T P\mathcal{A} & -\varepsilon A_d^T P\mathcal{A}_1\\ 0_{n\times 2n} & 0_{n\times 2n} & 0_{n\times 4n}\end{bmatrix},\\
&\Theta^{(4)}_{\varepsilon,d_M}=\begin{bmatrix}\omega_3 & 0_{4n\times 2n} & \omega_4\\ * & 0_{2n\times 2n} & 0_{2n\times 4n}\\ * & * & \varepsilon\mathcal{A}_1^T P\mathcal{A}_1\end{bmatrix},\\
&\Theta^{(5)}_{\varepsilon,d_M}=\begin{bmatrix}\omega_5 & \omega_6 & \omega_7\\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 4n}\\ \varepsilon\mathcal{A}_1^T P\mathcal{A}(I_2\otimes A_d) & -\varepsilon\mathcal{A}_1^T P\mathcal{A} & \varepsilon\mathcal{A}_1^T P\mathcal{A}_1\end{bmatrix},\\
&\Theta^{(6)}_{\varepsilon,d_M}=\begin{bmatrix}\omega_8 & \omega_9 & \varepsilon[\mathcal{A}(I_2\otimes A_d)]^T P\mathcal{A}_1\\ * & \varepsilon\mathcal{A}^T P\mathcal{A} & -\varepsilon\mathcal{A}^T P\mathcal{A}_1\\ * & * & \varepsilon\mathcal{A}_1^T P\mathcal{A}_1\end{bmatrix},\\
&\omega_1=\varepsilon A_d^T P A_d-\alpha^{d_M}(S+R)+\Sigma_0^{(2)},\\
&\omega_2=-A_d^T P(I+\varepsilon A_{av})\tilde{\mathcal{A}}+\varepsilon A_d^T P\mathcal{A}_\rho,\\
&\omega_3=\varepsilon\mathcal{A}_\rho^T P\mathcal{A}_\rho-\tilde{\mathcal{A}}^T(I+\varepsilon A_{av})^T P\mathcal{A}_\rho\\
&\qquad-\mathcal{A}_\rho^T P(I+\varepsilon A_{av})\tilde{\mathcal{A}}+\tilde{\mathcal{A}}^T Q_\theta\tilde{\mathcal{A}},\\
&\omega_4=-\varepsilon\mathcal{A}_\rho^T P\mathcal{A}_1+\tilde{\mathcal{A}}^T(I+\varepsilon A_{av})^T P\mathcal{A}_1,\\
&\omega_5=-\varepsilon\mathcal{A}_\rho^T P\mathcal{A}(I_2\otimes A_d)\\
&\qquad+\tilde{\mathcal{A}}^T(I+\varepsilon A_{av})^T P\mathcal{A}(I_2\otimes A_d),\\
&\omega_6=\varepsilon\mathcal{A}_\rho^T P\mathcal{A}-\tilde{\mathcal{A}}^T(I+\varepsilon A_{av})^T P\mathcal{A},\\
&\omega_7=-\varepsilon\mathcal{A}_\rho^T P\mathcal{A}_1+\tilde{\mathcal{A}}^T(I+\varepsilon A_{av})^T P\mathcal{A}_1,\\
&\omega_8=\varepsilon[\mathcal{A}(I_2\otimes A_d)]^T P[\mathcal{A}(I_2\otimes A_d)],\\
&\quad\omega_9=-\varepsilon[\mathcal{A}(I_2\otimes A_d)]^T P\mathcal{A}.
\end{aligned}\tag{88}$$

Summarizing, we arrive at:

Theorem 3. *Consider the system* (72) *subject to Assumption 2, let* H_a*, and* $H_\rho, H_{\rho,a}$ *be given by* (22) *and* (51)*, respectively. Given positive tuning parameters* θ*,* d_M^**, and* ε^* *subject to* (11) *and* $\theta\varepsilon^*<1$*. Let there exist* $0<P,S,R\in\mathbb{R}^{n\times n}$*,* $G\in\mathbb{R}^{n\times n}$*, and positive diagonal matrices* $\Lambda_{\mathcal{Y}_a},\Lambda_{\mathcal{Z}_a},\Lambda_{\mathcal{Z}_\rho}\in\mathbb{R}^{2\times 2}$ *and* $\Lambda_{\mathcal{Y}_\rho},\Lambda_{\mathcal{Y}_{\rho,a}},\Lambda_{\mathcal{Z}_{\rho,a}}\in\mathbb{R}^{4\times 4}$ *such that LMIs* (2) *and* (87) *with notations* (25) *and* (88) *hold with* $\varepsilon=\varepsilon^*$ *and* $d_M=d_M^*$*. Then system* (72) *is exponentially stable with a decay rate* $\sqrt{1-\theta\varepsilon}$ *for all* $\varepsilon\in(0,\varepsilon^*]$ *and* $0\le d_k\le d_M^*$*,* $\forall k\in Z_+$*. Namely, there exists* $M>0$ *such that for all* $\varepsilon\in(0,\varepsilon^*]$ *and* $0\le d_k\le d_M$*, the solution of* (72) *initialized by* $\{x_j\}_{j=-d_M}^0$ *satisfies*

$$|x_k|^2\le M|x|^2_{[-d_M,0]}(1-\theta\varepsilon)^k,\ \forall k\in\mathbb{Z}_+. \tag{89}$$

Moreover, if LMIs (2) *and* (87) *hold with* $\varepsilon=\varepsilon^*$*,* $d_M=d_M^*$ *and* $\theta=0$*, then* (72) *is exponentially stable for all* $\varepsilon\in(0,\varepsilon^*]$ *and* $0\le d_M\le d_M^*$*. Also, given any* d_M*, the inequalities* (2) *and* (87) *are always feasible for small enough* ε *and* θ*.*

Proof. The fact that feasibility of (87) with θ, ε^*, d_M^* implies its feasibility for all $\varepsilon<\varepsilon^*$, $d_M<d_M^*$, follows from monotonicity of (87) with respect to $\varepsilon<\varepsilon^*$, $d_M<d_M^*$.

Feasibility of (87) implies that for all

$$V(k+1)-\alpha V(k)\le 0\Rightarrow V(k+1)\le\alpha^{k+1-d_M}V(d_M), \tag{90}$$

$$\begin{aligned}V(d_M)=&|z_{d_M}|_P^2+\textstyle\sum_{i=0}^{d_M-1}\alpha^{d_M-i-1}|x_i|_S^2\\ &+d_M\textstyle\sum_{i=-d_M}^{-1}\sum_{s=d_M+i}^{d_M-1}\alpha^{d_M-s-1}|x_{s+1}-x_s|_R^2.\end{aligned} \tag{91}$$

Also, $V(k)\ge\sigma_{min}(P)|z_k|^2$, for any $d_M\le k\in\mathbb{Z}_+$. Thus, there exists a constant $\bar{M}>0$ such that

$$|z_k|^2\le\bar{M}|x|^2_{[-d_M,0]}\alpha^{k-d_M},\ d_M\le k\in\mathbb{Z}_+. \tag{92}$$

From (14), (15) and (92), we find that

$$|x_k|^2\le\bar{M}\delta_1^2|x|^2_{[-d_M,0]}\alpha^{k-d_M}, \tag{93}$$

for all $k\ge d_M$. In addition, since x_k is bounded on the interval $[0,d]$, inequality (89) holds for all $k\in\mathbb{Z}_+$.

For the LMI feasibility guarantees for given d_M, we choose $\Lambda_{\mathcal{Y}_a}=\Lambda_{\mathcal{Z}_a}=\Lambda_{\mathcal{Z}_\rho}=\lambda I_2$, $\Lambda_{\mathcal{Y}_\rho}=\Lambda_{\mathcal{Y}_{\rho,a}}=\Lambda_{\mathcal{Z}_{\rho,a}}=\lambda I_4$, $R=\lambda I$, $S=\varepsilon I$ and $G=0$, where $\lambda>2\max\{\lambda_0^{(1)},\lambda_0^{(2)}\}$. Thus, LMI (2) is feasible. For $\theta=0$ (so $\alpha=1$), Θ_{ε,d_M} is independent of d_M whereas $\varepsilon^2 d_M^2\mathcal{L}^T R\mathcal{L}$ is small for small enough ε. So, it is sufficient to prove that $\Theta_{\varepsilon,d_M}<0$. Since A_{av} is Hurwitz matrix, there exists $0<P\in\mathbb{R}^n$ such that $\beta_1<0$ for small enough ε. It is easily seen that $\Psi_{\varepsilon,d_M}<0$ for large enough λ and small enough $\varepsilon>0$. Next, we apply Schur complement with respect to Θ_{ε,d_M}, whence $\Theta_{\varepsilon,d_M}<0$ iff $\beta_1-\frac{1}{\lambda}B(\lambda^{-1}\Psi_{\varepsilon,d_M})^{-1}B^T<0$. Note that $-(\lambda^{-1}\Psi_{\varepsilon,d_M})^{-1}$ is bounded as $\lambda\to\infty$ (converges to the identity matrix), whereas B and β_1 are independent of λ implying the feasibility of $\Theta_{\varepsilon,d_M}<0$. □

Numerical Example: (Stabilization by fast switching) Let

$$A_D(k)=\begin{cases}A_1,\ k\in[100n\varepsilon,100(n+0.4)\varepsilon),\\ A_2,\ k\in[100(n+0.4)\varepsilon,100(n+1)\varepsilon),\end{cases} \tag{94}$$

and

$$\begin{aligned}&A_1=10^{-1}\begin{bmatrix}-1 & 0.6\\ -0.6 & 0.4\end{bmatrix},\ A_2=10^{-1}\begin{bmatrix}0.2 & -0.4\\ 0.4 & 1\end{bmatrix},\\ &A_d=10^{-1}\begin{bmatrix}-0.28 & 0\\ 0 & -0.44\end{bmatrix},A_0=0.\end{aligned} \tag{95}$$

Here, $A_D(k)$ can be presented as (41) with A_0, A_1, A_2 and A_d given by (95) and

$$\begin{aligned}&a_1(k)=\begin{cases}0.6,\ k\in[100n\varepsilon,100(n+0.4)\varepsilon),\\ -0.4,\ k\in[100(n+0.4)\varepsilon,100(n+1)\varepsilon),\end{cases}\\ &a_2(k)=\begin{cases}-0.6,\ k\in[100n\varepsilon,100(n+0.4)\varepsilon),\\ 0.4,\ k\in[100(n+0.4)\varepsilon,100(n+1)\varepsilon),\end{cases}\end{aligned} \tag{96}$$

We consider $\theta\in\{0,0.01\}$, $\varepsilon\in\{0.05,0.1\}$. Note that with some simple calculations, it is easily shown that (11) holds for $\varepsilon=0.05$ and $\varepsilon=0.1$. We consider systems (40) and (72) with constant and time-varying delays respectively, and verify the LMIs of Theorems 2 and 3, to obtain the maximal values of d and d_M which preserve the stability. The results are given in Table II. Clearly smaller ε allows for larger delays. Moreover, for $d_M=153$ and $\theta=0$, we found that the LMIs of Theorem 3 are feasible for $\varepsilon^*=0.0075$, illustrating that the LMIs are feasible for large time-varying delays provided ε is small enough. Finally, simulations of solutions for $\varepsilon=0.05$ show stability for constant delays less or equal to 715 that illustrates conservatism of the LMI result with maximum delay 153.

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Method	θ	ε	
		0.05	0.1
Theorem 2 - Constant Delay - d^*	0	153	75
	0.01	135	66
Theorem 3 - Time-varying Delay - d_M^*	0	23	11
	0.01	19	9

Table II
MAXIMUM d^* AND d_M^* PRESERVING THE LMI FEASIBILITY.

IV. CONCLUSION

This paper develops a novel constructive approach to averaging for stability of discrete-time linear delayed systems with rapidly-varying periodic coefficients. We proposed a novel system presentation and state transformation that leads to a perturbed averaged system. Further, by employing a direct Lyapunov method, explicit LMI conditions for exponential stability were derived. The LMIs provide upper bounds on the small parameters that preserve exponential stability of the original system. Moreover, our LMIs are feasible for any bounded delay provided ε is small enough. The results may be further improvement in the future, and applied to various averaging-based control problems including extremum seeking.

REFERENCES

[1] H. An, Q. Wu, H. Xia, and C. Wang. Control of a time-varying hypersonic vehicle model subject to inlet un-start condition. *Journal of the Franklin Institute*, 355(10):4164–4197, 2018.
[2] F. Bullo. Averaging and vibrational control of mechanical systems. *SIAM Journal on Control and Optimization*, 41(2):542–562, 2002.
[3] B. Caiazzo, E. Fridman, and X. Yang. Averaging of systems with fast-varying coefficients and non-small delays with application to stabilization of affine systems via time-dependent switching. *Nonlinear Analysis: Hybrid Systems*, 48:101307, 2023.
[4] X. Cheng, Y. Tan, and I. Mareels. On robustness analysis of linear vibrational control systems. *Automatica*, 87:202–209, 2018.
[5] E. Fridman. *Introduction to time-delay systems: analysis and control*. Birkhauser, Systems and Control: Foundations and Applications, 2014.
[6] E. Fridman and J. Zhang. Averaging of linear systems with almost periodic coefficients: A time-delay approach. *Automatica*, 122:109287, 2020.
[7] C. J. Harris and M. JF. Stability of linear systems: some aspects of kinematic similarity. 1980.
[8] A. Jbara, R. Katz, and E. Fridman. Stability by averaging of linear discrete-time systems. In *2024 European Control Conference - Submitted*, 2024.
[9] R. Katz, F. Mazenc, and E. Fridman. Stability of rapidly time-varying systems via a delay free transformation. *Autmatica*, 2023.
[10] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.
[11] P. Li, J. Lam, R. Lu, and K.-W. Kwok. Stability and L_2 synthesis of a class of periodic piecewise time-varying systems. *IEEE Transactions on Automatic Control*, 64(8):3378–3384, 2018.
[12] F. Mazenc, M. Malisoff, and M. S. De Queiroz. Further results on strict Lyapunov functions for rapidly time-varying nonlinear systems. *Automatica*, 42(10):1663–1671, 2006.
[13] S. Meerkov. Principle of vibrational control: theory and applications. *IEEE Transactions on Automatic Control*, 25(4):755–762, 1980.
[14] P. G. Park, J. Ko, and C. Jeong. Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, 47:235–238, 2011.
[15] H. Sandberg and E. Möllerstedt. Periodic modelling of power systems. *IFAC Proceedings Volumes*, 34(12):89–94, 2001.
[16] C. R. Wang, Y. He, C. K. Zhang, W. H. Chen, and M. Wu. Delay-variation-dependent summation inequality and its application to stability analysis of discrete-time systems with time-varying delay. *Systems & Control Letters*, 184:105721, 2024.
[17] X. Xie and J. Lam. Guaranteed cost control of periodic piecewise linear time-delay systems. *Automatica*, 94:274–282, 2018.
[18] X. Yang, J. Zhang, and E. Fridman. Periodic averaging of discrete-time systems: A time-delay approach. *IEEE Transactions on Automatic Control*, 2022.
[19] J. Zhang and E. Fridman. L_2-gain analysis via time-delay approach to periodic averaging with stochastic extension. *Automatica*, 137:110126, 2022.
[20] X. M. Zhang, Q. L. Han, and X. Ge. A novel approach to H^∞ performance analysis of discrete-time networked systems subject to network-induced delays and malicious packet dropouts. *Automatica*, 136:110010, 2022.