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Using Delay for Control

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Abstract

This article reviews two techniques that use delay for control: time-delay approaches to control problems (which initially may be free of delays) and the intentional insertion of delays into the feedback. We begin with a now widely used time-delay approach to sampled-data control. In networked control systems with communication constraints, this is the only method that accommodates transmission delays larger than the sampling intervals. We present a predictor-based design that enlarges the maximum allowable delay, which is important for practical implementations. We then discuss methods that use artificial delays via simple Lyapunov functionals that lead to feasible linear matrix inequalities for small delays and simple sampled-data implementations. Finally, we briefly present a new time-delay approach—this time to averaging. Unlike previous results, this approach provides the first quantitative bounds on the small parameter, making averaging-based control (including vibrational and extremum-seeking control) reliable.

1. INTRODUCTION

What is your reaction when you hear the word delay—positive or negative? In this article, it will be positive. Time delays are omnipresent. They appear in the states, inputs, and outputs of dynamical systems. State delays are common in biology, medicine, chemistry, physics, economy, and finances (1). Input and output delays are inevitable in control systems. Usually small delays preserve properties of the delay-free system. For example, consider the system

$$\dot{x}(t) = -x(t - \mathrm{h}), \quad x(t) \in \mathbb{R}, \qquad 1.$$

with a constant delay $\mathrm{h} > 0$. For $\mathrm{h} = 0$, this system is positive (has positive solutions for positive initial conditions) and exponentially stable. For $0 < \mathrm{h} < \frac{1}{e}$, the positivity is preserved (2) [for initial conditions $x(0) > 0, x(s) = 0$ for $s < 0$] as well as the stability. For larger delays, $\frac{1}{e} < \mathrm{h} < \frac{\pi}{2}$, solutions of Equation 1 oscillate (change sign) but exponentially decay to zero. For $\mathrm{h} > \frac{\pi}{2}$, Equation 1 has unbounded and oscillating solutions (see **Figure 1**).

In some systems, even arbitrarily small delays can lead to instability. This may happen in partial differential equations (PDEs) (3). For example, consider the 1D wave equation

$$\begin{aligned} &z_{tt}(\xi,t) = z_{\xi\xi}(\xi,t), \quad \xi \in (0,1), \quad z(\xi,t) \in \mathbb{R}, \\ &z(0,t) = 0, \quad z_{\xi}(1,t) = -z_t(1,t-\mathrm{h}), \end{aligned} \qquad 2.$$

with a delay $\mathrm{h} > 0$ in the boundary damping $z_t(1, t - \mathrm{h})$. Here, z_t, z_ξ and z_{tt}, $z_{\xi\xi}$ denote first- and second-order partial derivatives in t and ξ, respectively. If $\mathrm{h} = 0$, then all solutions of Equation 2 are zero for $t \geq 2$, but for an arbitrary small $\mathrm{h} > 0$, the system has unbounded solutions. Another example is neutral-type systems (i.e., systems where the delay enters the highest-order derivative), where an arbitrary small delay can destabilize the system (see example 2.2 in Reference 4).

However, delays may be stabilizing, and one may artificially insert delays for controller design (5–12). For example, consider the following double integrator:

$$\ddot{x}(t) = u(t), \quad y(t) = x(t), \quad x(t) \in \mathbb{R}.$$

The system is not stabilizable by the nondelayed static output feedback $u(t) = K_0 y(t)$. Since $u(t) = \overline{K}_0 y(t) + \overline{K}_1 \dot{y}(t)$ stabilizes the system and $\dot{y}(t) \approx \frac{y(t)-y(t-\mathrm{h})}{\mathrm{h}}$, for a small enough h, the system is stabilizable by the delayed output feedback

$$u(t) = K_0 y(t) + K_1 y(t - \mathrm{h}),\ \mathrm{h} > 0,$$

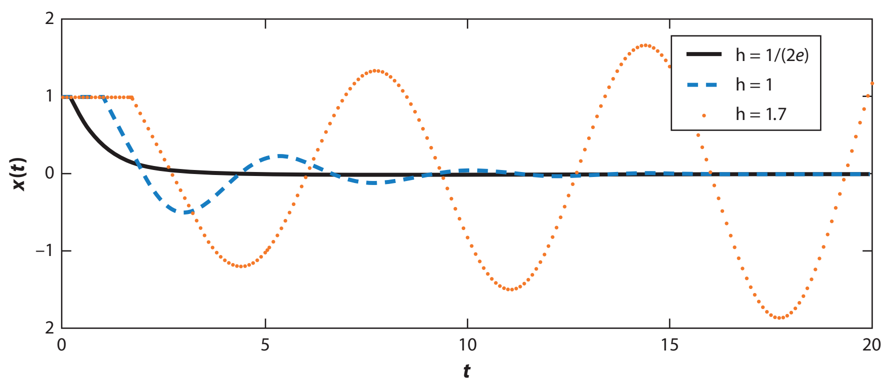

Figure 1

Solutions of Equation 1 for various values of h.

and by its sampled-data version, which is easy to implement:

$$u(t) = K_0 y(t_k) + K_1 y(t_{k-1}),\ t \in [t_k, t_{k+1}),\ t_{k+1} - t_k = \mathrm{h},\ k \in \mathbb{Z}_+ = \{0, 1, 2 \ldots\}.$$

Keeping in mind the abovementioned effects of delays on stability, for time-delay systems (TDSs), we aim to develop delay-robust design methods with constructive tools for finding an upper bound on the delays that preserve the stability and performance [i.e., exponential stability, input-to-state stability (ISS), and induced L_2 gain]. As in systems without delay, an efficient method for the stability analysis of TDSs is the Lyapunov method. For TDSs, there are two main Lyapunov methods: the Krasovskii method of Lyapunov functionals (13) and the Razumikhin method of Lyapunov functions (14). The two Lyapunov methods for linear TDSs can be combined with linear matrix inequalities (LMIs).

We review two time-delay approaches: a widely used one for sampled-data and networked control systems (NCSs) (Sections 2 and 3) and a recent one for averaging of systems with highly oscillating coefficients and averaging-based control (Section 5). Both approaches consist of two steps: (*a*) time-delay modeling of the closed-loop system and (*b*) stability and performance analysis. We also present simple Lyapunov-based methods for using artificial delay for stabilization (Section 4). Most of the results are presented for linear systems but may be extended to nonlinear ones, and we provide references for some of the nonlinear extensions.

2. A TIME-DELAY APPROACH TO SAMPLED-DATA CONTROL

Modern control systems usually employ digital technology for controller implementation, i.e., sampled-data control. Consider the linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^{n_u}, \qquad 3.$$

where $x(t)$ is the state and $u(t)$ is the control input. The control signal is assumed to be generated by the zero-order hold (ZOH) function $u(t) = u_{\mathrm{d}}(t_k)$, $t_k \leq t < t_{k+1}$, with a sequence of hold times

$$0 = t_0 < t_1 < \cdots < t_k < \cdots, \quad \lim_{k\to\infty} t_k = \infty, \qquad 4.$$

where u_{d} is a discrete-time control signal. The sampling interval can be either constant, with $t_{k+1} - t_k \equiv \mathrm{h}$, or variable, with $t_{k+1} - t_k \leq \mathrm{h}$. In the context of NCSs, the sampling interval may be variable and uncertain due to packet dropouts.

We consider a state-feedback control law $u(t) = Kx(t_k)$, $t_k \leq t < t_{k+1}$. The closed-loop sampled-data system has the form

$$\dot{x}(t) = Ax(t) + BKx(t_k), \quad t \in [t_k, t_{k+1}), \qquad 5.$$

and depends on both continuous and discrete time. There are three main approaches to sampled-data control that have become popular for NCSs (see 4, chapter 7; 15; 16): discrete-time, impulsive (or hybrid) system, and time-delay approaches. The discrete-time approach loses knowledge about intersampling behavior, which is especially important for the performance analysis of disturbed or nonlinear systems, whereas the hybrid and time-delay approaches preserve the intersampling information.

In the time-delay approach, the sampled-data system in Equation 5 is modeled as a continuous-time system with a time-varying delay $\tau(t) = t - t_k, t \in [t_k, t_{k+1})$ (17–19):

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)), \quad t \geq 0. \qquad 6.$$

Here, $\tau(t)$ is the time-varying (sawtooth) piecewise-linear delay (see **Figure 2**), which is upper-bounded by a known bound h (upper bound on the sampling intervals).

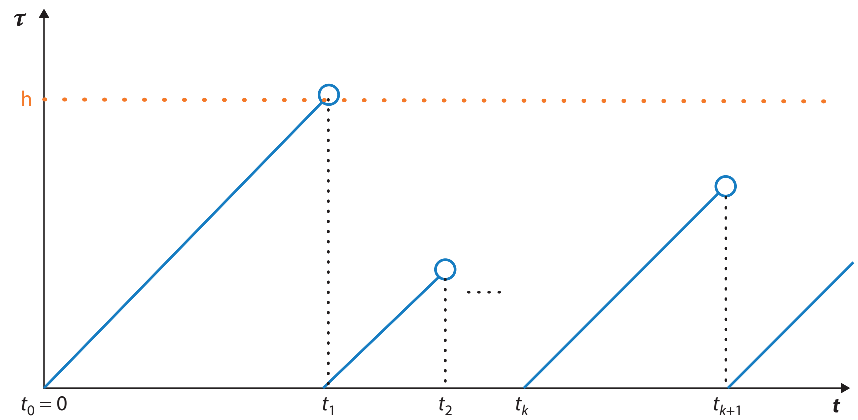

Figure 2

A sawtooth delay, $\tau(t)$.

2.1. Lyapunov-Based Stability and Performance Analysis for Time-Delay Systems

The second step in the time-delay approach to sampled-data control is the stability analysis of the model with a sawtooth delay, which may be accomplished by either the Lyapunov method or the input–output approach (see section 4.4 in Reference 4). Robust control of sampled-data systems was started in Reference 19 via Lyapunov functionals of Reference 20 for systems with fast-varying delays (without any constraints on the delay derivative). Note that, in slowly varying delays, it is assumed that $\dot{\tau} \leq d < 1$ with some constant d. In this section, we present detailed stability analysis for systems with fast-varying and sawtooth delays via the simplest Lyapunov functionals, whereas for advanced results we give references.

Consider the linear TDS

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad t \geq t_0, \tag{7.}$$

where $x(t) \in \mathbb{R}^n$ and $\tau(t) \in [0, \mathrm{h}]$ is a bounded fast-varying delay. Here, A and A_1 are constant $n \times n$ matrices. We are interested in simple delay-dependent (i.e., depending on h) stability conditions under the assumption that, for $\mathrm{h} = 0$, the system in Equation 7 is stable, i.e., the matrix $A + A_1$ is Hurwitz. The first delay-dependent (both Krasovskii- and Razumikhin-based) conditions were derived using the relation $x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s$, which transformed Equation 7 into

$$\dot{x}(t) = [A + A_1]x(t) - A_1 \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s \tag{8.}$$

via different model transformations (21, 22) [e.g., by substituting the right-hand side of Equation 7 for $\dot{x}(s)$, which brings significant conservatism (23, 24)].

We aim to construct a scalar Lyapunov functional candidate that is positive, $V \geq \alpha_1 |x(t)|^2$, $\alpha_1 > 0$, and whose derivative in time along Equation 8 is negative, $\dot{V} \leq -\alpha_2 |x(t)|^2$, $\alpha_2 > 0$. The latter two conditions guarantee the asymptotic stability of Equation 7 (for the corresponding Lyapunov–Krasovskii theorems, see, e.g., chapter 3 in Reference 4). We start with the Lyapunov function $V_P(x(t)) = x^{\mathrm{T}}(t)Px(t)$, where $0 < P \in \mathbb{R}^{n \times n}$, which corresponds to the nominal system $\dot{x}(t) = (A + A_1)x(t)$. Differentiating V_P along Equation 8, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V_P(x(t)) = 2x^{\mathrm{T}}(t)P\dot{x}(t) = 2x^{\mathrm{T}}(t)P\left[(A + A_1)x(t) - A_1 \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right]. \tag{9.}$$

To guarantee the negativity of $\dot{V}$, we have to add an additional term to V_P that leads (after differentiation and upper-bounding) to a negative quadratic term in $\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s$. Note that using Jensen's inequality (25, proposition B.8), we have

$$\left(\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} R \left(\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right) \leq \tau(t) \int_{t-\tau(t)}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \leq \mathrm{h} \int_{t-\mathrm{h}}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s, \qquad 10.$$

where $0 < R \in \mathbb{R}^{n\times n}$. This motivates the following functional introduced in Reference 20 for fast-varying delays:

$$V_R(\dot{x}_t) = \int_{-\mathrm{h}}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta = \int_{t-\mathrm{h}}^{t} (\mathrm{h}+s-t)\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s, \quad 0 < R \in \mathbb{R}^{n\times n}. \qquad 11.$$

Note that V_R depends on $\dot{x}_t := \dot{x}(t+\theta),\ \theta \in [-\mathrm{h}, 0]$. Its derivative,

$$\begin{aligned}\dot{V}_R(\dot{x}_t) &= \frac{\mathrm{d}}{\mathrm{d}t} V_R(\dot{x}_t) = \mathrm{h}\dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - \int_{t-\mathrm{h}}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \\ &\overset{\text{Eq. 10}}{\leq} \mathrm{h}\dot{x}^{\mathrm{T}}(t)R\dot{x}(t) - \frac{1}{\mathrm{h}}\left(\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}} R \left(\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right),\end{aligned} \qquad 12.$$

introduces a negative term that compensates the cross terms with $\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s$. It also adds a (small for small h) positive term. The functional $V = V_P + V_R$ can be used to study the stability of Equation 8. A simple LMI for the negativity of $\dot{V}$ follows from the summation of Equation 9 with Equation 12 and further application of the Schur complement lemma to $\mathrm{h}\dot{x}^{\mathrm{T}}(t)R\dot{x}(t)$ with $\dot{x}$ replaced by the right-hand side of Equation 8. This LMI is always feasible for small h.

Another way to derive LMI stability conditions via the same V is to use the descriptor method (26), which comes from the descriptor model transformation of Equation 8 in the form of the descriptor system $\dot{x}(t) = y(t)$ and $y(t) = (A+A_1)x(t) - A_1 \int_{t-\tau(t)}^{t} y(s)\mathrm{d}s$. The system in Equation 7 with a fast-varying delay was analyzed for the first time using the Krasovskii method in Reference 20 via the descriptor method, where $\dot{x}(t)$ is not replaced with the right-hand side of the differential equation. Instead, introducing the slack variables $P_2, P_3 \in \mathbb{R}^{n\times n}$, one has

$$\begin{aligned}\dot{V}_P &= 2x^{\mathrm{T}}(t)P\dot{x}(t) = 2x^{\mathrm{T}}(t)P\dot{x}(t) \\ &\quad + 2\left[x^{\mathrm{T}}(t)P_2^{\mathrm{T}} + \dot{x}(t)^{\mathrm{T}}(t)P_3^{\mathrm{T}}\right]\left[-\dot{x}(t) + (A+A_1)x(t) - A_1\int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right]\end{aligned} \qquad 13.$$

since the second line of Equation 13 is zero due to Equation 8. Then,

$$\dot{V} = \dot{V}_P + \dot{V}_R \overset{\text{Eq. 12}}{\leq} \eta^{\mathrm{T}}\Psi\eta \leq -\alpha(|x(t)|^2 + |\dot{x}(t)|^2),\ \alpha > 0,$$

with $\eta(t) = \mathrm{col}\{x(t), \dot{x}(t), \frac{1}{\mathrm{h}}\int_{t-\tau}^{t} \dot{x}(s)\mathrm{d}s\}$, if

$$\Psi = \begin{bmatrix} P_2^{\mathrm{T}}(A+A_1) + (A+A_1)^{\mathrm{T}}P_2 & P - P_2^{\mathrm{T}} + (A+A_1)^{\mathrm{T}}P_3 & -\mathrm{h}P_2^{\mathrm{T}}A_1 \\ * & -P_3 - P_3^{\mathrm{T}} + \mathrm{h}R & -\mathrm{h}P_3^{\mathrm{T}}A_1 \\ * & * & -\mathrm{h}R \end{bmatrix} < 0. \qquad 14.$$

Here, $*$ denotes symmetric terms in the symmetric matrix.

The descriptor method brought the free weighting matrices P_2 and P_3 into the Lyapunov analysis with the following benefits: less conservative conditions (even without delay) for uncertain systems; simple design LMIs for systems with state, input, and output delays by choosing $P_3 = \varepsilon P_2$ with a tuning scalar parameter ε (see Reference 27 and chapter 5 in Reference 4); unifying LMIs for the discrete-time and continuous-time systems having almost the same form (chapter 6 in Reference 4); and efficient LMI stability conditions for parabolic PDEs (28–30). Note that similar slack matrices can be introduced by using Finsler's lemma (31, 32).

To reduce the conservatism of the stability conditions via $V = V_P + V_R$, the relation between $x(t-\tau(t))$ and $x(t-\mathrm{h})$ [and not only between $x(t-\tau(t))$ and $x(t)$] was taken into account in Reference 33 by adding to V the additional term V_S:

$$V(x_t, \dot{x}_t) = x^{\mathrm{T}}(t)Px(t) + \mathrm{h}V_R(\dot{x}_t) + V_S(x_t), \quad V_S(x_t) = \int_{t-\mathrm{h}}^{t} x^{\mathrm{T}}(s)Sx(s)\mathrm{d}s, \ 0 < S \in \mathbb{R}^{n\times n}. \qquad 15.$$

Further improvement was achieved in Reference 34, where for upper-bounding of $\dot{V}_R$ (given by the first line of Equation 12) the reciprocally convex inequality was suggested with some matrix $G \in \mathbb{R}^{n\times n}$ (after application of Jensen's inequality):

$$\begin{aligned} -\mathrm{h}\int_{t-\mathrm{h}}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s &= -\mathrm{h}\int_{t-\mathrm{h}}^{t-\tau(t)} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s - \mathrm{h}\int_{t-\tau(t)}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s \\ &\leq -\frac{\mathrm{h}}{\tau(t)}e_1^{\mathrm{T}}Re_1 - \frac{\mathrm{h}}{\mathrm{h}-\tau(t)}e_2^{\mathrm{T}}Re_2 \leq -\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R & G \\ G^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad \begin{bmatrix} R & G \\ G^{\mathrm{T}} & R \end{bmatrix} > 0. \end{aligned} \qquad 16.$$

Here, $e_1 = x(t) - x(t-\tau(t))$ and $e_2 = x(t-\tau(t)) - x(t-\mathrm{h})$.

For the exponential stability, to satisfy the condition $\dot{V} + 2\alpha V \leq 0$ with $\alpha > 0$, one can insert $e^{-2\alpha(t-s)}$ into the integral terms of V (see details in chapter 4 of Reference 4):

$$V = x^{\mathrm{T}}(t)Px(t) + \int_{t-\mathrm{h}}^{t} e^{-2\alpha(t-s)}x^{\mathrm{T}}(s)Sx(s)\mathrm{d}s + \mathrm{h}\int_{-\mathrm{h}}^{0}\int_{t+\theta}^{t} e^{-2\alpha(t-s)}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta. \qquad 17.$$

The same V from Equation 17 can be used for the ISS analysis of the perturbed system in Equation 7.

The above conditions guarantee the stability for a small delay, $\tau(t) \in [0, \mathrm{h}]$. Many applications motivate the stability analysis for an interval (or nonsmall) delay $\tau(t) \in [\mathrm{h}, \mathrm{h}_1]$ with $\mathrm{h} > 0$. Keeping in mind that Equation 7 can be represented as

$$\dot{x}(t) = Ax(t) + A_1x(t-\mathrm{h}) - A_1\int_{t-\tau(t)}^{t-\mathrm{h}} \dot{x}(s)\mathrm{d}s,$$

one can analyze the stability of Equation 7 via the Lyapunov functionals of the form $V(x_t, \dot{x}_t) = V_{\mathrm{n}}(x_t, \dot{x}_t) + V_1(x_t, \dot{x}_t)$ (35), where V_{n} is the nominal functional for the exponentially stable nominal system with constant delay $\dot{x}(t) = Ax(t) + A_1x(t-\mathrm{h})$, and where

$$V_1 = \int_{t-\mathrm{h}_1}^{t-\mathrm{h}} x^{\mathrm{T}}(s)S_1x(s)\mathrm{d}s + (\mathrm{h}_1 - \mathrm{h})\int_{-\mathrm{h}_1}^{-\mathrm{h}}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta, \ S_1 > 0, \quad R_1 > 0. \qquad 18.$$

For the recent improvements of the stability conditions for systems with fast-varying delay by using augmented Lyapunov functionals, Bessel–Legendre inequalities, and novel reciprocally convex inequalities, we refer readers to References 36–38 and the references therein. The improved conditions are usually formulated in terms of higher-order LMIs. Note that, in contrast to constant delays, where Lyapunov functionals may depend only on x_t, the existing Lyapunov functionals for fast-varying delays depend on $\dot{x}_t$. The latter may be challenging for stochastic (where $\dot{x}$ is not well defined) or PDE systems. In the stochastic case, the deterministic part of $\dot{x}_t$ may be employed together with additional terms for Lyapunov functional constructions (39).

2.2. Time-Dependent Lyapunov Functionals for Systems with Sawtooth Delays

Consider Equation 7 with the sawtooth delay $\tau(t) = t - t_k, t \in [t_k, t_{k+1}), t_{k+1} - t_k \leq \mathrm{h}$. Before Reference 40 was published, the conventional time-independent Lyapunov functionals $V(x_t, \dot{x}_t)$ for systems with fast-varying delays had been applied to sampled-data systems (19). In

Reference 40, the sawtooth evolution of the delays was taken into account via the introduction of time-dependent Lyapunov functionals [inspired by the impulsive system approach to sampled-data control (41)]. Instead of the $V_R(\dot{x}_t)$ given by Equation 11, the following term was suggested:

$$V_R(t,\dot{x}_t) = (t_{k+1}-t)\int_{t_k}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s,\ t \in [t_k, t_{k+1}), \quad R > 0, \tag{19}$$

which continuously depends on time and after the differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}V_R(t,\dot{x}_t) = -\int_{t_k}^{t} \dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s + (t_{k+1}-t)\dot{x}^{\mathrm{T}}(t)R\dot{x}(t)$$

gives the same negative term as in Equation 12 but a smaller positive term. The time-dependent $V = x^{\mathrm{T}}(t)Px(t) + V_R(t,\dot{x}_t)$ via the descriptor method and Jensen's inequality leads to an LMI, which is affine in $t_{k+1}-t$ and $t-t_k$, whose feasibility is guaranteed if the resulting LMIs in two vertices ($t \to t_k$ and $t \to t_{k+1}$) hold:

$$\begin{bmatrix} \Phi_{11} & P-P_2^{\mathrm{T}}+(A+A_1)^{\mathrm{T}}P_3 \\ * & -P_3-P_3^{\mathrm{T}}+\mathrm{h}R \end{bmatrix} < 0, \quad \begin{bmatrix} \Phi_{11} & P-P_2^{\mathrm{T}}+(A+A_1)^{\mathrm{T}}P_3 & -\mathrm{h}P_2^{\mathrm{T}}A_1 \\ * & -P_3-P_3^{\mathrm{T}} & -\mathrm{h}P_3^{\mathrm{T}}A_1 \\ * & * & -\mathrm{h}R \end{bmatrix} < 0, \tag{20}$$

where $\Phi_{11} = P_2^{\mathrm{T}}(A+A_1) + (A+A_1)^{\mathrm{T}}P_2$. It can be seen that the LMIs in Equation 20 are less restrictive than the LMI in Equation 14 (leading to an h in Example 1 below that is twice as large).

Reference 42 suggested a different time-dependent (but discontinuous in time) Lyapunov functional that is based on the vector extension of Wirtinger's inequality:

$$\int_a^b z^{\mathrm{T}}(\xi)Wz(\xi)\mathrm{d}\xi \le \frac{4(b-a)^2}{\pi^2}\int_a^b \dot{z}^{\mathrm{T}}(\xi)W\dot{z}(\xi)\mathrm{d}\xi, \quad W > 0, \tag{21}$$

which holds for absolutely continuous $z : (a,b) \to \mathbb{R}^n$ with $\dot{z} \in L^2(a,b)$ and $z(a) = 0$. The Lyapunov functional $V = x^{\mathrm{T}}(t)Px(t) + V_W(t,x_t,\dot{x}_t)$ $(P, W > 0)$ with nonnegative (due to Wirtinger's inequality) term

$$\begin{aligned} &V_W(t,x_t,\dot{x}_t) = \mathrm{h}^2\int_{t_k}^{t} \dot{x}^{\mathrm{T}}(s)W\dot{x}(s)\mathrm{d}s - \frac{\pi^2}{4}\int_{t_k}^{t}[x(s)-x(t_k)]^{\mathrm{T}}W[x(s)-x(t_k)]\mathrm{d}s, \\ &W > 0, \quad t_k \le t < t_{k+1}, \quad k \in \mathbb{Z}_+, \end{aligned} \tag{22}$$

is discontinuous in time at the points $t = t_k$ but does not grow at these points. Thus, $\dot{V} \le -\alpha_2|x(t)|^2$ with some $\alpha_2 > 0$ guarantees the stability of Equation 7 with the sawtooth delay. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}V_W = \mathrm{h}^2\dot{x}^{\mathrm{T}}(t)W\dot{x}(t) - \frac{\pi^2}{4}[x(t_k)-x(t)]^{\mathrm{T}}W[x(t_k)-x(t)],$$

we directly arrive at the following stability condition (which recovers the result of Reference 43 derived via the small-gain theorem):

$$\begin{bmatrix} P(A+A_1)+(A+A_1)^{\mathrm{T}}P & PA_1 & \mathrm{h}(A+A_1)^{\mathrm{T}}W \\ * & -\frac{\pi^2}{4}W & \mathrm{h}A_1^{\mathrm{T}}W \\ * & * & -W \end{bmatrix} < 0, \quad P > 0,\ W > 0. \tag{23}$$

The Wirtinger-based LMI in Equation 23 is a single LMI with fewer decision variables than Equation 20. In contrast to the Lyapunov functional in Equation 19 (and its extensions, e.g., in References 40 and 44), the Wirtinger-based functionals can be efficiently applied to interval sawtooth delays (42). For the exponential or ISS stability, V_W can be modified as follows (45):

$$V_W = \mathrm{h}^2 e^{2\alpha\mathrm{h}}\int_{t_k}^{t} e^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)W\dot{x}(s)\mathrm{d}s - \frac{\pi^2}{4}\int_{t_k}^{t} e^{2\alpha(s-t)}[x(s)-x(t_k)]^{\mathrm{T}}W[x(s)-x(t_k)]\mathrm{d}s,$$

where $\alpha > 0$ and $V_W \geq 0$ due to the extended Wirtinger's inequality (see lemma 4 in Reference 45).

The time-dependent Lyapunov functional constructions discussed above have become efficient for various sampled-data and networked-based control problems, including NCSs with network-induced delays (46) and NCSs with scheduling protocols (47–49).

Example 1. Consider the scalar system

$$\dot{x}(t) = -x(t - \tau(t)), \tag{24}$$

which is asymptotically stable for a constant delay $\tau < \pi/2$, for all fast-varying $\tau(t) < 1.5$, and for sawtooth $\tau(t) = t - t_k < 2, t \in [t_k, t_{k+1}), k \in \mathbb{Z}_+$, and unstable otherwise. The LMI in Equation 14, the conditions in Reference 34, and the conditions in Reference 38 guarantee asymptotic stability for all fast-varying delays with upper bounds of 0.99, 1.33, and 1.40, respectively. By the impulsive system approach (41), the Wirtinger-based LMI in Equation 23, and the LMIs in Equation 20, the upper bounds on the variable samplings are 1.28, 1.57, and 1.99, respectively (the latter two bounds cannot be achieved by treating the sawtooth delays as fast-varying, where the upper bound is less than 1.5).

3. A TIME-DELAY APPROACH TO NETWORKED CONTROL SYSTEMS

NCSs are systems with spatially distributed sensors, actuators, and controller nodes that exchange data via a communication network. Among the benefits of NCSs are their ease of installation and maintenance, flexibility, low cost, and long-distance estimation and control. However, for the design of NCSs, the following undesirable perturbations should be taken into account: variable sampling or transmission intervals, variable communication delays, packet dropouts caused by the unreliability of the network, quantization errors, and communication constraints (where only one sensor or actuator node is allowed to transmit its packet per transmission, and where scheduling protocols are needed to orchestrate the transmission order of the nodes) (50).

Consider the static output-feedback control of an NCS shown in **Figure 3** with the linear plant defined by Equation 3 and the output $y = Cx \in \mathbb{R}^{n_y}$. Here, we have two networks (from sensor to controller and from controller to actuator) with separate sensor and actuator nodes. The sampler on the sensor side is time driven, whereas the controller and the ZOH are event driven (in the sense that the controller and the ZOH update their outputs as soon as they receive a new sample).

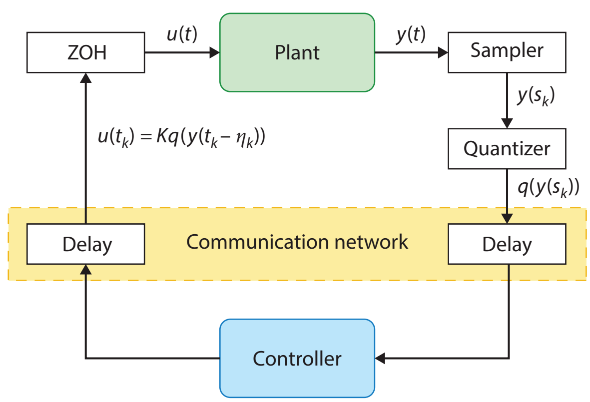

Figure 3

A networked control system with static output feedback. Abbreviation: ZOH, zero-order hold.

We assume that the output $y(s_k)$ is available at discrete sampling instants

$$0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad \lim_{k\to\infty} s_k = \infty, \qquad 25.$$

and its quantized values, $q(y(s_k))$, are transmitted via the networks. The quantizers are piecewise-constant functions $q\colon \mathbb{R}^{n_y} \to \mathcal{D}$, where $\mathcal{D}$ is a countable or finite subset of $\mathbb{R}^{n_y}$. There are several types of quantizers, e.g., uniform (51) and logarithmic (52). Consider, e.g., a uniform quantizer, where the quantization error is upper-bounded by a known constant $\Delta > 0 : |q(z) - z| \leq \Delta \ \forall z \in \mathbb{R}^{n_y}$ [for the case where $|q(z) - z| \leq \Delta$ holds for $|z| \leq M$ and $|q(z)| > M - \Delta$ for $|z| > M$ (here, $M > 0$ is the quantization range), see References 53 and 54].

We take into account data packet dropouts by allowing the sampling to be nonuniform. In our formulation, $y(s_k)$ corresponds to the measurements that are not lost. Denote by t_k the updating instant time of the ZOH, and suppose that the updating signal at the instant t_k has experienced a signal transmission delay η_k. As in References 46 and 55, we allow the delays η_k to be larger than the sampling intervals $s_{k+1} - s_k$, provided that the sequence of input update times t_k remain strictly increasing. This means that if an old sample gets to the destination after the most recent one, it should be dropped.

The static output-feedback controller implemented via ZOH has the form

$$u(t) = Kq(y(t_k - \eta_k)), \quad t_k \leq t < t_{k+1}, \quad k = 0, 1, 2, \ldots, \qquad 26.$$

with t_{k+1} being the next updating instant time of the ZOH after t_k.

To extend the time-delay approach from sampled-data control to NCS, define

$$\tau(t) = t - t_k + \eta_k, \quad t_k \leq t < t_{k+1}. \qquad 27.$$

Assume that

$$t_{k+1} - t_k + \eta_k \leq \tau_{\mathrm{M}}, \quad k \in \mathbb{Z}_+, \qquad 28.$$

where τ_{M} denotes the maximum time span between the time $s_k = t_k - \eta_k$ at which the state is sampled and the time t_{k+1} at which the next update arrives at the ZOH. Suppose that the network-induced delay is lower-bounded: $\eta_k \geq \eta_{\mathrm{m}}$, where $\eta_{\mathrm{m}} \geq 0$ is a known bound. We have a bounded piecewise-linear delay

$$\eta_{\mathrm{m}} \leq \tau(t) = t - t_k + \eta_k \leq t_{k+1} - t_k + \eta_k \leq \tau_{\mathrm{M}} \quad \forall k \in \mathbb{Z}_+$$

with $\dot{\tau}(t) = 1$ for $t \neq t_k$. Let MAD (maximum allowable delay) be a known upper bound on the network-induced delay η_k and MATI be the maximum allowable transmission interval, meaning that successive sensor measurements are separated by at most MATI seconds (56). Assume that $\delta \in \mathbb{Z}_+$ is a bound on the maximum number of successive dropouts. Then, $\tau_{\mathrm{M}} = (1 + \delta)\mathrm{MATI} + \mathrm{MAD}$.

We obtain the following time-delay model of the closed-loop system:

$$\dot{x}(t) = Ax(t) + BKCx(t - \tau(t)) + BKw(t), \quad t \geq 0, \qquad 29.$$

where $w(t) = q(y(s_k)) - y(s_k)$ is the quantization error and $|w(t)| \leq \Delta$. ISS analysis of the system in Equation 29 with a time-varying delay $\tau(t) \in [\eta_{\mathrm{m}}, \tau_{\mathrm{M}}]$ can be provided by deriving conditions for $\dot{V} + 2\alpha V - \gamma^2|w|^2 \leq 0 \ \forall w$ (with some constants $\alpha > 0$ and γ^2) along Equation 29 via an appropriate Lyapunov functional V (see the sidebar titled Networked Control Systems Under Dynamic Quantizers).

NETWORKED CONTROL SYSTEMS UNDER DYNAMIC QUANTIZERS

The time-delay approach can be efficiently applied to NCSs under dynamic quantizers with the zoom strategy introduced in Reference 51, which is composed of zooming out (the range of quantizer is increased to guarantee that output can be adequately measured) and zooming in (the quantization error is decreased to drive the states to the origin). To derive constructive conditions for finding the zooming times, Lyapunov-based bounds on the solutions of the delayed closed-loop system and additional bounds on the solutions of the open-loop system $\dot{x}(t) = Ax(t)$ for $t \geq 0$ such that $t - \tau(t) < 0$ are employed in Reference 54. For the time-delay approach to NCSs under the logarithmic quantizer, where the quantization error is presented as a sector-bounded nonlinearity, we direct readers to Reference 57.

3.1. Networked Control Systems Under Scheduling Protocols

The communication constraints impose that, for each transmission, only one node can access the network and send its information. Hence, communication along the data channel is orchestrated by a scheduling rule called a protocol. There are three main classes of network protocols: (*a*) static protocols, including the round-robin protocol, where the signals from nodes are transmitted periodically one after another (the order is decided in advance) (50, 58, 59); (*b*) dynamic protocols, which include the well-known try-once-discard (TOD) protocol (47, 49, 50, 58, 60), where the node that has the largest deviation of the current value of the signal from the latest transmitted one is granted access to the network; and (*c*) the stochastic protocol, which determines the transmitted node through a Bernoulli or a Markov chain process with a known probability for collisions (48, 61).

Three main approaches have been applied to NCSs in the presence of scheduling protocols: the hybrid approach (50, 58), the discrete-time approach (60), and the time-delay approach (47–49, 59). So far, only the time-delay approach has allowed for large communication delays (larger than the sampling intervals) in the presence of scheduling protocols.

Consider the LTI system in Equation 3 with N measurements $y_i = C_i x$, $i = 1, \ldots, N$, in the presence of scheduling protocols from sensors to controller, as shown in **Figure 4** under the static output-feedback controller with a gain $K = [K_1, \ldots, K_N]$ such that $A + \sum_{i=1}^{N} BK_iC_i$ is Hurwitz. For

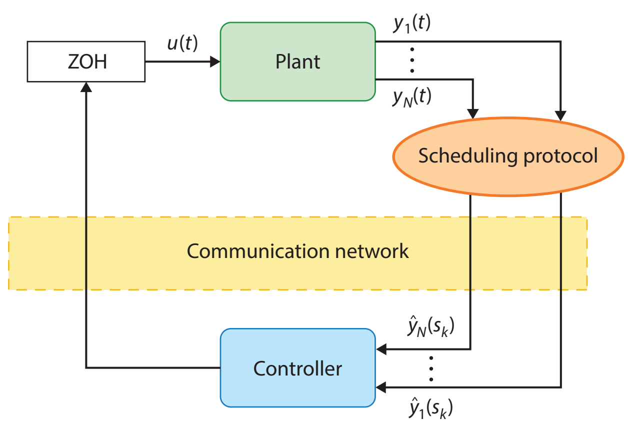

Figure 4
Scheduling from sensors to controller in a networked control system. Abbreviation: ZOH, zero-order hold.

the round-robin protocol, under the assumptions on the sampling instants and network-induced delays as in the previous section (without quantization and packet dropouts), the simplest closed-loop model is a model with multiple delays:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} BK_iC_ix(t - \tau_i(t)), \quad t \geq t_{N-1}, \tag{30.}$$

where $\eta_m \leq \tau_i(t) \leq N \cdot \text{MATI} + \text{MAD}$ (49). The stability of the latter system can be studied by using Lyapunov functionals for interval delays. Reference 59 presented a more accurate (and complicated) switched-system model of the closed-loop system with ordered delays that may improve the results (but increase the complexity of the conditions). Under the TOD and stochastic protocols, the time-delay approach leads to the impulsive model with delays in the continuous dynamics and in the reset equations, where discontinuous Lyapunov functionals are useful (for the extension of the time-delay approach under scheduling protocols to stochastic systems, see Reference 62). In the numerical examples via the time-delay approach, the round-robin protocol usually leads to a larger MATI and MAD (and essentially improves the results via other approaches).

Reference 49 extended the time-delay approach to decentralized NCSs with multiple independent local communication networks under the round-robin or TOD protocol of coupled systems:

$$\dot{x}_j = A_jx_j + \sum_{l\neq j} F_{lj}x_l + B_ju_j, \quad y_j = C_jx_j, \quad j = 1, \ldots, M, \tag{31.}$$

where $x_j \in \mathbb{R}^{n_j}$, $y_j \in \mathbb{R}^{p_j}$, and F_{lj} is a coupling matrix. The network-based static output feedback $u_j = K_jy_j$ was considered. Efficient (for weak coupling with comparatively small $\|F_{lj}\|$) stability conditions for the coupled system were derived by using Lyapunov functionals V_j with nonintegral parts $x_j^{\mathrm{T}}P_jx_j,\ P_j > 0$ (appropriate for each closed-loop j subsystem), by verifying the conditions

$$\dot{V}_j(t) + 2\alpha V_j(t) \leq \frac{2\varepsilon}{M-1} \sum_{l\neq j} x_l^{\mathrm{T}}(t)P_lx_l(t) \quad \forall j = 1, \ldots, M \tag{32.}$$

for some $\alpha > \varepsilon > 0$. Indeed, the latter inequality guarantees the exponential stability of the entire system since its Lyapunov functional $V = \sum_{j=1}^{M} V_j$ satisfies $\dot{V} + 2(\alpha - \varepsilon)V \leq 0$. For detailed reviews on the time-delay approach to NCSs, we direct readers to References 16 and 63.

3.2. Compensation of Large Constant Parts of Delays in Networked Control Systems Using Predictors

To further enlarge communication (and additional input/output) delays that preserve NCS performance, predictors can be used for compensation of large and known constant parts of delays. There are three main predictor methods: (*a*) classical predictors, which extended Smith's predictor to unstable systems using the state-space representation (64); (*b*) PDE-based predictors (65); and (*c*) sequential subpredictors (66). Classical predictors via a reduction approach (67) and subpredictors can efficiently compensate known constant parts of fast-varying nonsmall delays that appear in NCSs. We briefly present the two latter methods below. Note that it is difficult to apply the PDE-based predictor in the presence of (additional to constant) fast-varying delay (68), but the PDE predictor has been extended to unknown constant delays (69, 70).

3.2.1. Classical predictors and the reduction approach.

Consider the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t - \mathrm{h}),\ t \geq 0, \qquad x : [0, \infty) \to \mathbb{R}^n, \quad u : [-\mathrm{h}, \infty) \to \mathbb{R}^m, \tag{33.}$$

with a constant input delay $\mathrm{h} > 0$ and $u(t) = 0$ for $t < 0$. If there is $K \in \mathbb{R}^{m\times n}$ rendering $A + BK$ Hurwitz, the control $u(t - \mathrm{h}) = Kx(t)$ would exponentially stabilize Equation 33. The problem is that $u(t) = Kx(t + \mathrm{h})$ depends on the future state value, which cannot be measured. The idea of the classical predictor is to predict $x(t + \mathrm{h})$ by using a variation-of-constants formula for Equation 33 on $[t, t + \mathrm{h}]$:

$$x(t + \mathrm{h}) = e^{A\mathrm{h}}x(t) + \int_t^{t+\mathrm{h}} e^{A(t+\mathrm{h}-s)}Bu(s - \mathrm{h})\mathrm{d}s \Rightarrow$$

$$u(t) = K\left[e^{A\mathrm{h}}x(t) + \int_{t-\mathrm{h}}^{t} e^{A(t-s)}Bu(s)\mathrm{d}\xi\right], \quad t \geq 0.$$

The latter is a causal controller depending on the past values of $u(s)$ with $s \in [t - \mathrm{h}, t]$.

For the control of systems with uncertain input delay (as appears in NCSs),

$$\dot{x}(t) = Ax(t) + Bu(t - \mathrm{h} - \eta(t)), \quad t \geq 0, \quad 0 \leq \eta(t) \leq \mu < \mathrm{h}, \qquad 34.$$

where h is known, the model reduction approach is efficient (67, 71). In this approach, the change of variables

$$z(t) = e^{A\mathrm{h}}x(t) + \int_{t-\mathrm{h}}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s$$

reduces Equation 34 with $u(t) = Kz(t)$ to [nondelayed for $\eta(t) = 0$] the system

$$\dot{z}(t) = (A + BK)z(t) + e^{A\mathrm{h}}BK[z(t - \mathrm{h} - \eta(t)) - z(t - \mathrm{h})].$$

The stability of this system can be analyzed using the Lyapunov functional $V = x^{\mathrm{T}}(t)Px(t) + V_1$, $P > 0$, where V_1 is defined by Equation 18 with $\mathrm{h}_1 = \mathrm{h} + \mu$ (68, 72).

For the output-feedback control of Equation 33 with $y(t) = Cx(t)$, the Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) - LC(x(t) - \hat{x}(t)) + Bu(t - \mathrm{h}) \qquad 35.$$

is employed with L such that $A + LC$ is Hurwitz. The observer-based predictor has the form

$$u(t) = K\hat{z}(t), \quad \hat{z}(t) = e^{A\mathrm{h}}\hat{x}(t) + \int_{t-\mathrm{h}}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s. \qquad 36.$$

The stability of the resulting closed-loop system can be analyzed by separating the stability analysis of the estimation error $e(t) = x(t) - \hat{x}(t)$, which satisfies $\dot{e}(t) = (A + LC)e(t)$, and the stability analysis of

$$\dot{\hat{z}}(t) = A\hat{z}(t) + Bu(t) - e^{A\mathrm{h}}LCe(t) = (A + BK)\hat{z}(t) - e^{A\mathrm{h}}LCe(t). \qquad 37.$$

Note that, if the input and output constant delays are $\mathrm{h}_u \geq 0$ and $\mathrm{h}_y > 0$, respectively, then the output delay can be moved to the input, with the resulting input delay $\mathrm{h} = \mathrm{h}_u + \mathrm{h}_y$.

For the output-feedback control of Equation 34 with a known h and $y(t) = Cx(t)$, the same predictor given by Equations 35 and 36 can be used, leading to the coupled closed-loop system in Equation 37 and

$$\dot{e}(t) = (A + LC)e(t) + BK(\hat{z}(t - \mathrm{h} - \eta(t)) - \hat{z}(t - \mathrm{h})).$$

The stability conditions for the latter system can be derived by using an appropriate Lyapunov functional. In the presence of two networks (from sensors to controllers and controllers to actuators), it can be assumed that the resulting measurement delay $\tau_y(t)$ is known (the measurements are sent together with the time stamps). Then the observer in Equation 35 can be applied with the innovation term $LC\left(x(t - \tau_y(t)) - \hat{x}(t - \tau_y(t))\right)$ [instead of $LC\,(x(t) - \hat{x}(t))$] together with the predictor in Equation 36. For classical predictors in application to NCSs, we direct readers to References 45 and 73.

3.2.2. Sequential subpredictors. Consider Equation 33 with $y(t) = Cx(t)$. The main idea of the subpredictor feedback is to construct an observer (subpredictor) $\hat{x}(t)$ of the future state $x(t + \mathrm{h})$ (66):

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L[C\hat{x}(t-\mathrm{h}) - y(t)]$$

with the feedback $u(t) = K\hat{x}(t)$. Here, L and K are chosen to make $A + LC$ and $A + BK$ Hurwitz. The prediction error $e(t) = \hat{x}(t-\mathrm{h}) - x(t)$ is governed by

$$\dot{e}(t) = Ae(t) + LCe(t-\mathrm{h}).$$

If the value of h is too large and leads to instability, the large delay h is divided into small fractions h/M, $M \in \mathbb{N}$, with sequential subpredictors (66, 74)

$$\hat{x}_1(t) \approx x(t+\mathrm{h}), \quad \hat{x}_2(t) \approx x\left(t + \frac{M-1}{M}\mathrm{h}\right), \quad \ldots, \quad \hat{x}_M(t) \approx x\left(t + \frac{\mathrm{h}}{M}\right),$$

governed by

$$\begin{aligned}
\dot{\hat{x}}_1(t) &= A\hat{x}_1(t) + LC\left(\hat{x}_1\left(t - \tfrac{\mathrm{h}}{M}\right) - \hat{x}_2(t)\right) + Bu(t),\\
\dot{\hat{x}}_2(t) &= A\hat{x}_2(t) + LC\left(\hat{x}_2\left(t - \tfrac{\mathrm{h}}{M}\right) - \hat{x}_3(t)\right) + Bu\big(t - \tfrac{\mathrm{h}}{M}\big),\\
&\vdots\\
\dot{\hat{x}}_M(t) &= A\hat{x}_M(t) + L\left(C\hat{x}_M\left(t - \tfrac{\mathrm{h}}{M}\right) - y(t)\right) + Bu\big(t - \tfrac{M-1}{M}\mathrm{h}\big).
\end{aligned} \qquad 38.$$

The errors $e_k(t) = \hat{x}_k(t-\mathrm{h}/M) - \hat{x}_{k+1}(t)$ $(k = 1, \ldots, M-1)$, $e_M(t) = \hat{x}_M(t-\mathrm{h}/M) - x(t)$, satisfy

$$\begin{aligned}
\dot{e}_k(t) &= Ae_k(t) + LCe_k(t-\mathrm{h}/M) - LCe_{k+1}(t), \quad k = 1, \ldots, M-1,\\
\dot{e}_M(t) &= Ae_M(t) + LCe_M(t-\mathrm{h}/M).
\end{aligned} \qquad 39.$$

The system described by Equation 39 is stable provided its last equation is stable (i.e., M is large enough). The feedback $u(t) = K\hat{x}_1(t)$ stabilizes Equation 33 since the closed-loop system given by Equation 39 and

$$\dot{\hat{x}}_1(t) = (A + BK)\hat{x}_1(t) + LCe_1(t) \qquad 40.$$

with $e_k \to 0$ $(k = 1, \ldots, M)$ is stable.

If h is known and $\eta(t)$ is unknown in Equation 34, then the subpredictors have the same form as Equation 38 with the control law $u(t) = K\hat{x}_1(t)$. In this case, Equations 39 and 40 cannot be separated, because the last equation of Equation 39 is replaced with

$$\dot{e}_M(t) = Ae_M(t) + LCe_M\left(t - \frac{\mathrm{h}}{M}\right) + BK\left(\hat{x}_1(t-\mathrm{h}) - \hat{x}_1(t-\mathrm{h}-\eta(t))\right). \qquad 41.$$

Here, the stability analysis of the coupled system (modified Equations 39 and 40) should be provided. (For more on comparisons of predictors and subpredictors, see the sidebar titled Classical Predictors Versus Subpredictors.)

CLASSICAL PREDICTORS VERSUS SUBPREDICTORS

Subpredictors can be easily applied to systems with norm-bounded uncertainties and nonlinearities. For the comparison of the classical predictor and subpredictors in application to decentralized control of large-scale systems with independent networks, we direct readers to Reference 75, where subpredictors lead to a larger constant part of delay and delay uncertainty in the examples. Subpredictors are also efficient in the presence of a round-robin protocol from sensors to controllers (75).

3.3. A Time-Delay Approach to Event-Triggered Control

The event-triggering mechanism (ETM) is used to reduce the number of signals to be transmitted through a communication network (76, 77). When the time-delay approach with an appropriate Lyapunov functional is applicable to NCSs, the ETM may further reduce the number of sent signals, though the benefits are usually demonstrated only by numerical simulations (the workload reduction has been analytically proven only for stochastic systems, e.g., in Reference 78). Two main time-delay approaches to event-triggered control have been suggested: the discrete-time ETM (which includes periodic as a particular case) (79, 80) and the continuous-time ETM with a dwell time (the switched-system approach) (81).

Consider an LTI system with the output $y(t) \in \mathbb{R}^{n_y}$ being transmitted through a communication channel at the sampling instants t_k, $k \in \mathbb{Z}_+$. Then, the event-triggering condition is checked periodically at the discrete times $t_k + i\mathrm{h}$ with $i = 1, 2, \ldots$ (79, 80) and

$$t_{k+1} = \min\{t_k + i\mathrm{h} \mid |\sqrt{\Omega}(y(t_k + i\mathrm{h}) - y(t_k))|^2 > \sigma|\sqrt{\Omega}y(t_k + i\mathrm{h})|^2\}, \tag{42}$$

whereas an ETM with a dwell time $\mathrm{h} > 0$ is checked continuously (81):

$$t_{k+1} = \min\{t \geq t_k + \mathrm{h} \mid |\sqrt{\Omega}(y(t) - y(t_k))|^2 \geq \sigma|\sqrt{\Omega}y(t)|^2\}. \tag{43}$$

Here, $t_0 = 0$, $\sigma > 0$, and $0 < \Omega \in \mathbb{R}^{n_y \times n_y}$. The ETMs in Equations 42 and 43 guarantee that the interevent time is at least h, which rules out the Zeno behavior. The ETM in Equation 43 can be viewed as a switching between the time-triggered sampling t_k with $t_{k+1} - t_k \leq \mathrm{h}$ and the continuous ETM, bringing advantages in the stability analysis compared with the discrete-time ETM (81). Both ETMs based on the time-delay approach to NCSs can easily treat additional network-induced delays. A switched-system approach with continuous-time ETM verification appeared to be natural for observer-based control, where the control law is computed continuously in time, but the ETM is used in the network from the controller to the actuator (82). For a review on time-delay approach to event-triggered control, we direct readers to Reference 16.

3.4. Network-Based Control of Partial Differential Equations and Multiagent Deployment

Network-based long-distance control of chemical reactors or areas with air pollution that can be modeled by diffusion PDEs (83, 84) is potentially of great interest. A time-delay approach to sampled-data and network-based control of parabolic PDEs has been suggested for distributed (in the spatial domain) or boundary control of parabolic PDEs via two methods: spatial decomposition (29, 85) and modal decomposition (82, 86–88). In the modal decomposition (applicable to distributed and boundary control) (89), the controller is constructed on the basis of a finite-dimensional system that captures the dominant dynamics of the infinite-dimensional one. In the spatial decomposition (applicable to distributed control), the spatial domain is divided into n subdomains with n sensors and actuators located in each subdomain. Finite-dimensional output-feedback controllers [static via spatial decomposition (29) and observer based via modal decomposition (87)] appear to be robust with respect to delays. Observer-based controllers are efficient for delay compensation (88). Delayed (boundary or in-domain) measurements can be analyzed by combining the Lyapunov functionals and Halanay's inequality (29, 90, 91).

The ETMs in Equations 42 and 43 for the heat PDE were presented in References 92 and 82, respectively. Network-based H_∞ filtering of the N-dimensional heat PDE in the presence of the round-robin protocol was studied in Reference 85. Sampled-data control via time-delay approach was suggested for the heat (93), Kuramoto–Sivashinsky (94), damped wave, and beam (95, 96) equations (see the sidebar titled Partial Differential Equation–Based Deployment).

PARTIAL DIFFERENTIAL EQUATION–BASED DEPLOYMENT

Deployment of a multiagent system, where a group of agents rearrange their positions into a target spatial configuration in order to achieve a common goal, has attracted the attention of many researchers (97). When the number of agents is large, a methodology based on PDEs becomes efficient (98, 99). A PDE-based approach to multiple agents has the advantage of inherent scalability, bringing new, efficient, PDE-inspired decentralized control laws. In the case of measurements of the leaders' absolute positions, the majority of PDE-based results employ the PDE observer, which may be difficult to implement. References 95 and 100 suggested a simple static output-feedback controller where the leaders' absolute positions were transmitted to other agents by using a communication network. In References 101 and 102, leaders employed network-based boundary control and avoided communicating with other agents (which may be expensive and not secure).

4. USING ARTIFICIAL DELAYS FOR CONTROL

It is well known that time delay is, in many cases, a source of instability. However, for some systems, the presence of delay can have a stabilizing effect (5–7, 103). Thus, some classes of systems (e.g., chains of integrators, oscillators, or inverted pendulums) that cannot be stabilized by memoryless static output feedback can be stabilized by using static output feedback with delays (8, 9, 104, 105). The idea of feedback design in this case is usually based on the employment of a stabilizing feedback that depends on the output derivatives (which can hardly be measured directly) and further approximation of the output derivatives by finite differences. Similarly, the output derivative in a proportional–integral–derivative (PID) controller can be replaced by its Euler approximation (10).

It is clear that the methods that assume the stability of the delay-free system are not applicable to the delay-induced stability, where delay has a stabilizing effect. One way to obtain stability conditions for stabilizing delays is to use frequency domain analysis, which gives necessary and sufficient conditions in the case of LTI systems (6, 106). This approach, however, is challenging to use with high-order systems, multiple delays, or nonlinear dynamics. Alternatively, one can use discretized Lyapunov functionals (25) or augmented Lyapunov functionals (107). These approaches lead to higher-order LMIs without feasibility guarantees. Static output-feedback controllers with stabilizing artificial delays are attractive due to their simplicity in implementation, and the first simple and efficient LMI conditions with feasibility guarantees for the design and robustness analysis of such controllers were suggested in References 11 and 12. Here, we present two approaches that lead to simple LMIs guaranteeing delay-induced stability with feasibility guarantees.

4.1. Lyapunov–Krasovskii Analysis via Neutral-Type Transformation

Consider the following second-order system:

$$\ddot{y}(t) = A_0 y(t) + A_1 \dot{y}(t) + Bu(t), \quad y \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \qquad 44.$$

Such systems are ubiquitous in mechanical engineering and describe, e.g., an inverted pendulum on a cart. This system can be rewritten as

$$\dot{x}(t) = \begin{bmatrix} 0 & I_n \\ A_0 & A_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t), \qquad x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}. \qquad 45.$$

Assume that Equation 44 is stabilizable, i.e., there exists $\bar{K} \in \mathbb{R}^{m\times n}$ such that the state feedback

$$u_{\heartsuit}(t) = \bar{K}x(t) = \bar{K}_0 y(t) + \bar{K}_1 \dot{y}(t) \qquad 46.$$

stabilizes the system. However, the output derivative, $\dot{y}$, is either impossible or difficult to measure in practice. Instead, it can be approximated using the backward finite difference $\dot{y} \approx \frac{y(t)-y(t-\mathrm{h})}{\mathrm{h}}$, giving rise to the time-delay feedback:

$$u_{\heartsuit}(t) \approx u(t) = \bar{K}_0 y(t) + \bar{K}_1 \frac{y(t) - y(t-\mathrm{h})}{\mathrm{h}} = \left[\bar{K}_0 + \frac{\bar{K}_1}{\mathrm{h}}\right] y(t) - \frac{\bar{K}_1}{\mathrm{h}} y(t-\mathrm{h}).$$

The resulting delayed static output feedback has the form

$$u(t) = K_0 y(t) + K_1 y(t-\mathrm{h}), \qquad K_0 = \bar{K}_0 + \bar{K}_1/\mathrm{h}, \quad K_1 = -\bar{K}_1/\mathrm{h}. \tag{47.}$$

To find an upper bound on h that guarantees the stability of the closed-loop system given by Equations 45 and 47, we use the following representation:

$$y(t-\mathrm{h}) = y(t) - \mathrm{h}\dot{y}(t) + \frac{\mathrm{d}}{\mathrm{d}t} G(\dot{y}_t), \qquad G(\dot{y}_t) = \int_{t-\mathrm{h}}^{t} (s - t + \mathrm{h}) \dot{y}(s) \mathrm{d}s. \tag{48.}$$

Then, Equation 45 under Equation 47 can be presented as a neutral system (since G depends on the past value of the state $\dot{y}_t = x_{1t}$) with a Hurwitz matrix D:

$$\dot{z}(t) = Dx(t), \qquad z = x(t) - \begin{bmatrix} 0 \\ B \end{bmatrix} K_1 G(\dot{y}_t), \qquad D = \begin{bmatrix} 0 & I_n \\ A_0 + B\bar{K}_0 & A_1 + B\bar{K}_1 \end{bmatrix}. \tag{49.}$$

The following Lyapunov functional can be used for the stability analysis of Equation 49:

$$V(x_t) = z^{\mathrm{T}}(t) P z(t) + V_R, \; V_R = \mathrm{h}^2 \int_{t-\mathrm{h}}^{t} (s - t + \mathrm{h})^2 \dot{y}^{\mathrm{T}}(s) K_1^{\mathrm{T}} R K_1 \dot{y}(s) \mathrm{d}s, \quad P > 0, \; R > 0.$$

Using an extended Jensen's inequality, one can show that $V(x_t) \geq \alpha |x(t)|^2$ for some $\alpha > 0$ (39). The term V_R compensates for G in the Lyapunov analysis since the extended Jensen's inequality gives

$$\dot{V}_R \leq \mathrm{h}^2 \dot{y}^{\mathrm{T}}(t) \bar{K}_1^{\mathrm{T}} R \bar{K}_1 \dot{y}(t) - 4 G^{\mathrm{T}}(\dot{y}_t) K_1^{\mathrm{T}} R K_1 G(\dot{y}_t).$$

This leads to

$$\dot{V} \leq \begin{bmatrix} x(t) \\ -K_1 G(\dot{y}_t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} D^{\mathrm{T}} P + PD + \mathrm{diag}\left\{0, \mathrm{h}^2 \bar{K}_1^{\mathrm{T}} R \bar{K}_1\right\} & D^{\mathrm{T}} P [0 \;\; B^{\mathrm{T}}]^{\mathrm{T}} \\ * & -4R \end{bmatrix} \begin{bmatrix} x(t) \\ -K_1 G(\dot{y}_t) \end{bmatrix}, \tag{50.}$$

which gives the desired stability LMI. Moreover, this LMI is always feasible for a small enough h. The above method with V depending on $\dot{y}$ and not on $\ddot{y}$ (as presented in Section 4.2) can be directly extended to stochastic systems (11).

4.2. The Direct Lyapunov–Krasovskii Approach and Sampled-Data Implementation

Another way to express the derivative approximation error is to use Taylor's expansion with the remainder in the integral form (which holds if $\dot{y}$ is absolutely continuous):

$$\begin{aligned} & y(t-\mathrm{h}) = y(t) - \dot{y}(t)\mathrm{h} + \int_{t-\mathrm{h}}^{t} (s - t + \mathrm{h}) \ddot{y}(s) \mathrm{d}s \quad \Rightarrow \\ & \frac{y(t) - y(t-\mathrm{h})}{\mathrm{h}} = \dot{y}(t) - r(t), \qquad r(t) = \frac{1}{h} \int_{t-\mathrm{h}}^{t} (s - t + \mathrm{h}) \ddot{y}(s) \mathrm{d}s. \end{aligned} \tag{51.}$$

Using this representation, one can rewrite the closed-loop system in Equations 45 and 47 as

$$\dot{x}(t) = Dx(t) + \begin{bmatrix} 0 \\ B \end{bmatrix} K_1 r(t), \tag{52.}$$

with D given in Equation 49 and $r(t)$ defined above. A suitable Lyapunov functional has the form

$$V = x^{\mathrm{T}}Px + V_R, \quad V_R = \int_{t-\mathrm{h}}^{t} (s-t+\mathrm{h})^2 \dddot{y}^{\mathrm{T}}(s)K_1^{\mathrm{T}}RK_1\dddot{y}(s)\mathrm{d}s, \qquad P>0, \quad R>0. \qquad 53.$$

By the extended Jensen's inequality (4, lemma 4.7), we have

$$\dot{V}_R \le \mathrm{h}^2\dddot{y}^{\mathrm{T}}(t)K_1^{\mathrm{T}}RK_1\dddot{y}(t) - 4r^{\mathrm{T}}(t)K_1^{\mathrm{T}}RK_1r(t). \qquad 54.$$

Then, differentiating V along Equation 52 and applying the Schur complement, we arrive at the following LMI, which is always feasible for small h:

$$\begin{bmatrix} D^{\mathrm{T}}P + PD & P[0 \ \ B^{\mathrm{T}}]^{\mathrm{T}} & [(A_0 + B\bar{K}_0),\ (A_1 + B\bar{K}_1)]^{\mathrm{T}}\mathrm{h}^2\bar{K}_1^{\mathrm{T}}R\bar{K}_1 \\ * & -4R & \mathrm{h}^2B^{\mathrm{T}}\bar{K}_1^{\mathrm{T}}R\bar{K}_1 \\ * & * & -\mathrm{h}^2\bar{K}_1^{\mathrm{T}}R\bar{K}_1 \end{bmatrix} < 0. \qquad 55.$$

For the sampled-data delayed implementation of Equation 46, the approximations

$$y(t) \approx y(t_k), \qquad \dot{y}(t) \approx \frac{y(t_k) - y(t_{k-1})}{\mathrm{h}}, \qquad t \in [t_k, t_{k+1}), \qquad t_{k+1} - t_k = \mathrm{h},$$

lead to the sampled-data delayed control $u(t) = K_0y(t_k) + K_1y(t_{k-1}), t \in [t_k, t_{k+1})$, with the gains defined in Equation 47 (108). LMIs for the stability of the resulting closed-loop system (which are always feasible for small h) have been derived in Reference 108.

Example 2. Consider the uncertain double integrator $\ddot{y}(t) = gy(t) + u(t)$ with an uncertain $g \in [-0.1, 0.1]$ under the discrete-time measurements $y(t_k), t \in [t_k, t_{k+1}), t_{k+1} - t_k = \mathrm{h}$. This system cannot be stabilized by $u(t) = Ky(t_k), t \in [t_k, t_{k+1})$, with any K. An observer-based sampled-data control for the uncertain system leads to complicated stability conditions and implementation. To design a simple static output feedback $u(t) = K_0y(t_k) + K_1y(t_{k-1})$, we choose Equation 46 with $\bar{K}_0 = -0.25$ and $\bar{K}_1 = -0.0499$. The approach proposed in this section leads to LMIs that are affine in g. By verifying them at the two vertices $g = \pm 0.1$, we find that the system is stable for $\mathrm{h} \in (0, 0.258]$. Taking $\mathrm{h} = 0.258$, we conclude, by using Equation 47, that the sampled-data controller with $K_0 = -0.4434$ and $K_1 = 0.1934$ exponentially stabilizes the system.

4.3. Extensions: Higher-Order Systems, PID, and Nonlinear and Adaptive Control

Consider an LTI system $\dot{x}(t) = Ax(t) + Bu(t),\ y = Cx(t)$ with relative degree $r \ge 2$, i.e.,

$$CB = CAB = \cdots = CA^{r-2}B = 0, \qquad CA^{r-1}B \ne 0.$$

Relative degree is how many times the output $y(t)$ needs to be differentiated before the input $u(t)$ appears explicitly. For such systems, it is common to look for a stabilizing controller depending on the ith-order derivatives ($i = 1, \ldots r-1$):

$$u_{\heartsuit}(t) = \bar{K}_0y(t) + \bar{K}_1y^{(1)}(t) + \cdots + \bar{K}_{r-1}y^{(r-1)}(t). \qquad 56.$$

However, usually the derivatives are not accessible. They may be approximated $\tilde{y}_i(t) \approx y^{(i)}(t)$, e.g., using finite differences:

$$\tilde{y}_0(t) = y(t),\ \tilde{y}_i(t) = \frac{\tilde{y}_{i-1}(t) - \tilde{y}_{i-1}(t-\mathrm{h})}{\mathrm{h}} = \frac{1}{\mathrm{h}^i}\sum_{k=0}^{i}\binom{i}{k}(-1)^k y(t-k\mathrm{h}), \quad i \in \mathbb{N}, \qquad 57.$$

with a delay $\mathrm{h} > 0$ and the binomial coefficients $\binom{i}{k} = \frac{i!}{k!(i-k)!}$. Substituting the latter into Equation 56, we obtain the time-delay implementation of the derivative-dependent control:

$$u(t) = \sum_{i=0}^{r-1}\bar{K}_i\tilde{y}_i(t) \overset{57}{=} \sum_{i=0}^{r-1}K_iy(t-i\mathrm{h}), \qquad K_i = (-1)^i\sum_{j=i}^{r-1}\binom{j}{i}\frac{1}{\mathrm{h}^j}\bar{K}_j, \quad i = 0, \ldots, r-1. \qquad 58.$$

For simple LMIs that guarantee the stability of the resulting closed-loop system as well as the delayed sampled-data implementation of Equation 56, we direct readers to Reference 109; a stochastic extension is given in Reference 110.

Finite-difference approximations of the derivative terms in the controller can be efficiently used for PID control, as suggested in Reference 10 for the second-order systems in continuous time; a sampled-data implementation is given in Reference 108, and an extension to an extended PID controller for higher-order systems is given in Reference 111. This approximation, leading to a delayed static controller, appeared to be efficient for nonlinear control (for sliding mode control, see References 112 and 113; for practical fixed-time stabilization, see References 114 and 115). Robust adaptive control by using delay under system uncertainty and measurement bias was presented in Reference 116. Local delay-induced ISS of nonlinear second-order systems was studied in Reference 117.

5. A TIME-DELAY APPROACH TO AVERAGING AND AVERAGING-BASED CONTROL

In control theory, the discovery in 1951 of the stabilizing effect of vibration in the inverted Stephenson–Kapitza pendulum (see 118) inspired the study of control by fast oscillations (119). This area became much more active (see, e.g., 120, 121) after the 1999 discovery that fast oscillating feedback gains can stabilize some systems in the cases where these systems cannot be stabilized by static ones [Brockett's problem (122)]. Many important control methods, including vibrational control (119), stabilization by fast switching, and extremum seeking (ES) (123), employ stabilization via fast oscillating signals (dithers), and the mathematical explanation of them is related to averaging (see, e.g., chapter 10 in Reference 124). The existing methods for stability analysis by averaging employ asymptotic analysis that is based on approximations (see, e.g., 125) and lead to important qualitative results: The system is stable provided that the averaged system is stable and the oscillations are fast enough. Thus, for the linear fast-varying system

$$\dot{x}(t) = A\left(\frac{t}{\varepsilon}\right)x(t), \quad x(t) \in \mathbb{R}^n, \quad t \geq 0, \quad \varepsilon > 0, \qquad 59.$$

with continuous 1-periodic A [i.e., $A(\tau + 1) = A(\tau)\ \forall\, \tau \geq 0$], the following holds (124): If the matrix $A_{\text{av}} = \int_0^1 A(\tau)\mathrm{d}\tau$ is Hurwitz, then Equation 59 is asymptotically stable for a small enough ε. But how can we find an efficient upper bound on ε that preserves the stability of Equation 59?

The first efficient upper bound on ε was suggested in Reference 126 via a time-delay approach that consisted of two steps: time-delay transformation of Equation 59 and Lyapunov–Krasovskii analysis. Consider Equation 59 with a piecewise-continuous, almost periodic A, meaning that $\int_t^{t+1} A(\tau)\mathrm{d}\tau = A_{\text{av}} + \Delta A(t)\ \forall t \geq 0$, where A_{av} is Hurwitz and $\sup_{t\geq 0}|\Delta A(t)|$ is small. Integrating Equation 59 for all $t \geq \varepsilon$, we have (cf. Equation 48)

$$\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\dot{x}(s)\mathrm{d}s = \frac{x(t)-x(t-\varepsilon)}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}t}[x(t)-G(t)], \qquad G(t) \triangleq \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\dot{x}(s)\mathrm{d}s. \qquad 60.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-G(t)] = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}A\left(\frac{s}{\varepsilon}\right)[x(s)+x(t)-x(t)]\mathrm{d}s,$$

where $\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}A(\frac{s}{\varepsilon})\mathrm{d}s = A_{\text{av}} + \Delta A$, leads to a neutral system in Hale's form:

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-G(t)] = (A_{\text{av}}+\Delta A)x(t) - Y(t), \quad Y(t) = \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}A\left(\frac{s}{\varepsilon}\right)\int_{s}^{t}\dot{x}(\theta)\mathrm{d}\theta\,\mathrm{d}s, \qquad 61.$$

with $\dot{x}(\theta) = A(\frac{\theta}{\varepsilon})x(\theta)$. If $x(t)$ satisfies Equation 59, then it also satisfies Equation 61. Thus, the stability of Equation 61 guarantees the stability of Equation 59. Note that $G = O(\varepsilon)$ and $Y = O(\varepsilon)$ for $x = O(1)$. Therefore, Equation 61 is a perturbation of the exponentially stable averaged system $\dot{x}(t) = (A_{\text{av}} + \Delta A)\,x(t)$. Then, appropriate Lyapunov functionals for Equation 61 lead to LMI conditions for finding an upper bound on ε (the length of the distributed delay) that preserves the exponential stability (126).

The time-delay approach to averaging leads to constructive averaging-based conditions for the ISS and L^2-gain analysis of the perturbed linear systems and to mean square stability of stochastic systems (127) with extensions to delayed, switched-affine, and discrete-time systems (128, 129). In Reference 130, this approach led to the first quantitative bounds on the frequency and amplitude of the fast-varying high gain of the static output feedback that stabilizes a class of linear systems (Brockett's problem). ISS with respect to the average value of disturbances for input-affine nonlinear systems was suggested in Reference 131.

In Reference 132, the time-delay approach has been applied to the model for constant power loads in single-phase AC systems $\dot{P}(t) = \frac{v^2(t)}{T}[u(t) - u(t-T)]$, where $P(t)$ is active power, $u(t)$ is control, and $v(t)$ is T-periodic voltage. The control objective here is to design a controller such that $\lim_{t\to\infty} P(t) = P_\star$ and $\lim_{t\to\infty} u(t) = u_\star$ with a given $P_\star > 0$ and arbitrary $u_\star > 0$. The PI controller with appropriate gains k_p and k_i achieves the goal:

$$u(t) = k_p\,(P_\star - P(t)) + k_i x_c(t), \quad \dot{x}_c(t) = P_\star - P(t).$$

5.1. Extremum Seeking via a Time-Delay Approach

ES is a powerful real-time optimization method that does not require knowledge of the system model. A majority of ES algorithms employ highly oscillating dithers, where the stability analysis is based on the qualitative averaging theory. In Reference 133, the first results for constructive ES of quadratic static maps were suggested via a time-delay approach. Consider, for simplicity, static single-input maps of quadratic form:

$$y(t) = f(\theta(t)) = f^* + \frac{f''}{2}\,(\theta(t) - \theta^*)^2,$$

where $y(t) \in \mathbb{R}$ is the output, and $\theta(t) \in \mathbb{R}$ is the input. Let the extremum value f^*, point θ^*, and Hessian $f'' \neq 0$ be constant. Usually f is unknown, but the sign of f'' is known. The objective of ES is to find a real-time estimate $\theta(t)$ of the extremum point θ^* ($\theta(t) \to \theta^*$) based on the measurements of $y(t)$. The classical ES algorithm has the form (134)

$$\theta(t) = \hat{\theta}(t) + a\sin(\omega t), \quad \dot{\hat{\theta}}(t) = ka\sin(\omega t)\cdot y(t), \tag{62}$$

where a and ω are the amplitude and frequency of the dither signal, and the adaptation gain k is such that $kf'' < 0$. The estimation error $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$ is governed by

$$\dot{\tilde{\theta}}(t) = ka\sin(\omega t)\left[f^* + \frac{f''}{2}\left(\tilde{\theta}(t) + a\sin(\omega t)\right)^2\right]. \tag{63}$$

Let $\omega = \frac{2\pi}{\varepsilon}$. The time-delay transformation in Equation 60 applied to Equation 63 leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{\theta}(t) - G(t)\right] = \frac{ka^2 f''}{2}\tilde{\theta}(t) - \frac{f''}{2}Y_1(t) - f''Y_2(t), \quad t \geq \varepsilon, \tag{64}$$

or, with further transformation $z = \tilde{\theta}(t) - G(t)$, suggested in Reference 135, to

$$\dot{z}(t) = \frac{ka^2 f''}{2}[z(t) + G(t)] - \frac{f''}{2}Y_1(t) - f''Y_2(t), \quad t \geq \varepsilon, \tag{65}$$

with the explicitly given expressions for the nonlinear $O(\varepsilon)$ terms G, Y_1, and Y_2 (133). Practical stability analysis of Equation 65 can be further provided via the Lyapunov–Krasovskii method or by employing the variation-of-constants formula (133, 135). In particular, for uncertain maps with f'' and f^* from known intervals and for any $\sigma_0 > 0$ and initial $|\tilde{\theta}(0)| < \sigma_0$, simple LMIs can be derived for finding the quantitative bounds on the dither frequency, the decay rate of the exponential convergence, and the ultimate bound on the estimation error.

The time-delay approach to ES was extended to N-dimensional quadratic maps and to bounded ES (introduced in Reference 136 and analyzed via Lie brackets approximations, but in the time-delay approach, the analysis works in the same manner as for classical ES) and to sampled-data and delayed implementation of ES (133, 135, 137). Note that in the environment without GPS orientation (136), where for the known maps the gradient is still unknown, the time-delay approach provides the first full solution with bounds on the suitable dithers, domains of attractions, and the resulting ultimate bounds. The time-delay approach to ES was extended to general nonlinear maps with a prior knowledge about the upper bounds of the nonlinear map and its gradient and Hessian (138), where the time-delay approach suggests a quantitative lower bound of dither frequency and the ultimate upper bound of estimation error, which is difficult to achieve by the classical averaging method.

A time-delay approach to a general input-affine system with the averaged system given by the Lie brackets system was presented in Reference 139 with an application to control of linear uncertain systems under unknown control directions using ES, as pioneered in Reference 140. Constructive LMIs were derived for finding upper bounds on the small parameter and measurement delay that ensure regional practical stability.

5.2. Concluding Remarks on Constructive Methods for Averaging

The time-delay approach to averaging provides approximation-free analysis, allows efficient quantitative and accurate qualitative bounds for averaging-based control, and offers tools for averaging of systems with delays, stochastic systems, and PDEs. It inspires new and more efficient constructive approaches (see, e.g., Reference 141, which includes delay-free transformation and a new presentation of Equation 59), leading to reliable control that employs averaging.

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