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# Delayed feedback implementation of decentralized derivative-dependent control of large-scale systems with input delays and disturbed measurements

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ABSTRACT

We study decentralized derivative-dependent control of large-scale $n$th-order systems with input delays via delayed feedback implementation. The unavailable derivatives can be approximated by finite differences giving rise to a time-delayed feedback. In the centralized case, an efficient simple linear matrix inequalities (LMIs)-based method for designing of such static output-feedback and its sampled-data implementation was recently suggested. In the present paper, we extend this design to large-scale systems in the presence of input delays and disturbed measurements. Under the assumption of the stabilizability of the system with small enough input delays and small enough interactions by a state-feedback that depends on the output and its derivatives, a delayed static output-feedback that stabilizes the system is presented by using the current and past disturbed measurements. To compensate the errors due to the input delays, we add the appropriate terms to the corresponding Lyapunov–Krasovskii functional that lead to LMIs conditions. The efficient bounds on the delays preserving that the resulting system is input-to-state stable (ISS) are found by verifying the LMIs. In addition, we employ the vector Lyapunov functional method that may allow larger couplings compared with the existing method. Finally, the effectiveness of the proposed methods is illustrated by numerical examples.

## 1. Introduction

During the past decades, much attention was paid to the control law that depends on the system output and its derivatives for the stabilization of linear systems. These derivatives are not available but can be approximated by the finite-difference method. The latter gives rise to a time-delayed feedback. The corresponding delay-induced stability was checked by frequency-domain techniques (Kharitonov, Niculescu, Moreno, & Michiels, 2005; Niculescu & Michiels, 2004; Ramírez, Mondié, Garrido, & Sipahi, 2015; Ramirez, Sipahi, Mondié, & Garrido, 2017), and complete Lyapunov–Krasovskii functionals (LKFs) (Egorov, 2016; Gu, Chen, & Kharitonov, 2003; Kharitonov, 2012), which presented necessary and sufficient conditions.

Simple LMIs for delay-induced stability were proposed in Fridman and Shaikhet (2016, 2019) and then extended to the $n$th-order systems in Fridman and Shaikhet (2017), Selivanov and Fridman (2018a). This method allows for performance and robustness analysis as well as stochastic perturbations (Zhang & Fridman, 2020). The key idea is to represent the delayed measurement by Taylor's expansion with a remainder, and further to compensate the latter by corresponding term in Lyapunov–Krasovskii functional. A remarkable improvement was presented in Selivanov and Fridman (2018b) for the $n$th-order system, where the derivative terms were expressed by finite differences with remainders and the controller implementation was, for the first time, presented by using consecutive sampling measurements. It was then extended to stochastic systems (Zhang & Fridman, 2020, 2022) and applied to platooning control of vehicular systems (Zhang, Peng, & Xie, 2023). Note that the controllers in the aforementioned work are of the centralized type.

In practical applications, e.g. power systems, communication networks, and aircraft engines (Guo, Hill, & Wang, 2000; Peng, Han, & Yue, 2012), the plant has a high dimensionality, information constraints, and distributed structure (Zhang & Lin, 2014), which is usually modeled as large-scale systems. In this case, centralized control may be not applicable to large-scale systems. As an efficient and effective way, decentralized control that uses locally available information of the subsystems only has received considerable attention with many important results (Baigzadehnoe, Rahmani, Khosravi, & Rezaie, 2020; Borgers & Heemels, 2014; Freirich & Fridman, 2016; Zhu & Fridman, 2020a). It should be noted that the design of the observer-based controller is very complicated (Liu, Xu, Xie, & Xiao, 2019; Yang & Dubljevic, 2014; Zhu & Fridman, 2020b). Thus, a simple static output-feedback is very attractive in the decentralized control of large-scale

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systems. In addition, input and output delays that may degrade the performance of the closed-loop and even lead to instability are non-negligible factors (Dolk, Borgers, & Heemels, 2016; Freirich & Fridman, 2016; Fridman & Shaked, 2005; Heemels, Borgers, van de Wouw, Nešić, & Teel, 2013). Moreover, in practice the measurement may be subject to unknown disturbances (Furtat, Fridman, & Fradkov, 2018; Liu, Wang, Zhang, Lu, & Kang, 2020; Sanz, Garcia, & Albertos, 2016; Xie, Tang, Song, Zhou, & Guo, 2018).

In this paper, we study decentralized derivative-dependent control of large-scale $n$th-order systems with known, constant input delays and bounded, $(n-1)$-times continuously differentiable measurement disturbances via delayed feedback implementation with disturbed measurements. Under the assumption of the stabilizability of the system with small enough input delays and interactions by a state-feedback that depends on the output and its derivatives, a delayed static output-feedback is presented using the current and past disturbed measurements. The latter leads to a closed-loop system with additional errors comparatively to that under the case of no input delay. To compensate these errors, we add the appropriate terms to the corresponding LKFs with efficient LMI conditions. In addition, inspired by Matrosov (1997) and Nersesov and Haddad (2006) we employ the vector Lyapunov functional method that may allow larger couplings compared with the existing method (Freirich & Fridman, 2016; Zhu & Fridman, 2020b). We suggest the sampled-data implementation by using consecutive sampling noisy measurements. Besides, we prove that the derived conditions are always feasible for small enough sampling period, input delays, and interactions if the continuous-time derivative-dependent feedback stabilizes the system. Finally, the effectiveness of the proposed methods is illustrated by numerical examples.

Summarizing, we have extended the efficient method (i.e. time-delay implementation of derivative-dependent feedback of Selivanov and Fridman (2018b)) from centralized to decentralized control, where we consider, for the first time, the measurements subject to the disturbance in the implementation. A conference version of this paper confined to the large-scale second-order systems without measurement disturbances was presented in Zhang, Zhang, Fridman and Peng (2023).

**Lemma 1** (*Solomon & Fridman, 2013*). *Let $\rho : [a,b] \to [0,\infty)$ and $f : [a,b] \to \mathbb{R}^n$ be such that the integration concerned is well-defined. Then for any $0 < R \in \mathbb{R}^{n\times n}$, the following holds:*

$$\int_a^b \rho(s) f^T(s) ds R \int_a^b \rho(s) f(s) ds \le \int_a^b \rho(s) ds \int_a^b \rho(s) f^T(s) R f(s) ds.$$

**Lemma 2** (*Selivanov & Fridman, 2016*). *Let $f : [a,b] \to \mathbb{R}^n$ be an absolutely continuous function with a square integrable first order derivative such that $f(a)=0$ or $f(b)=0$. Then*

$$\int_a^b e^{2\alpha t} f^T(s) W f(s) ds \le e^{2|\alpha|(b-a)} \frac{4(b-a)^2}{\pi^2} \int_a^b e^{2\alpha t} \dot f^T(s) W \dot f(s) ds$$

*for any $\alpha \in \mathbb{R}$ and $0 \le W \in \mathbb{R}^{n\times n}$.*

**Notations**: $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space with Euclidean norm $|\cdot|$, $\mathbb{R}^{n\times m}$ denotes the set of all $n\times m$ real matrices with the induced matrix norm $\|\cdot\|$. Denote by diag$\{\dots\}$ and col$\{\dots\}$ block-diagonal matrix and block-column vector, respectively. $P>0$ implies that $P$ is a positive definite symmetric matrix. Define $p \le\le q$ $(p \ge\ge q)$, where $p = \mathrm{col}\{p_1,\dots,p_n\}$ and $q = \mathrm{col}\{q_1,\dots,q_n\}$, if $p_i \le q_i$ $(p_i \ge q_i)$ for all $i = 1,\dots,n$.

## 2. Continuous delayed decentralized control

Consider the large-scale system composed of $M$ coupled plants $\mathcal{P}_j$ $(j=1,\dots,M)$, whose dynamics has the following form

$$\mathcal{P}_j :\ y_j^{(n)}(t) = \sum_{i=0}^{n-1} A_{ij} y_j^{(i)}(t) + B_j u_j(t-r_j) + \sum_{i=0}^{n-1} \sum_{l=1, l\ne j}^{M} F_{lji} y_l^{(i)} \quad t \ge 0, \tag{1}$$

where $\mathrm{col}\{y_j(t),\dots,y_j^{(n-1)}(t)\} \in \mathbb{R}^{nk_j}$ and $u_j(t) \in \mathbb{R}^{k_j^u}$ are the $j$th subsystem state and control input, $A_{ij} \in \mathbb{R}^{k_j\times k_j}$ and $B_j \in \mathbb{R}^{k_j\times k_j^u}$ are constant matrices, and $F_{lji} \in \mathbb{R}^{k_j\times k_j}$ are the interactions between plants $\mathcal{P}_j$ and $\mathcal{P}_l$. Without loss of generality, we assume $F_{jji}=0$ for $i=0,\dots,n-1$. The control input $u_j(t)$ is subject to a constant and known input delay $r_j>0$.

Denoting

$$x_j(t) = \mathrm{col}\{x_{0j}(t),\dots,x_{(n-1)j}(t)\} = \mathrm{col}\{y_j(t),\dots,y_j^{(n-1)}(t)\} \in \mathbb{R}^{nk_j},$$

$$A_j = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \\ A_{0j} & A_{1j} & A_{2j} & \dots & A_{(n-1)j} \end{bmatrix} \in \mathbb{R}^{nk_j\times nk_j},$$

$$\bar B_j = \mathrm{col}\{0, B_j\} \in \mathbb{R}^{nk_j\times k_j^u}, \quad F_{lj} = \mathrm{col}\{0, \bar F_{lj}\} \in \mathbb{R}^{nk_j\times nk_j},$$

$$\bar F_{lj} = \left[F_{lj0},\dots,F_{lj(n-1)}\right] \in \mathbb{R}^{k_j\times nk_j}, \tag{2}$$

system (1) is expressed by

$$\dot x_j(t) = A_j x_j(t) + \bar B_j u_j(t-r_j) + \sum_{l=1,l\ne j}^{M} F_{lj} x_l(t), \quad j=1,\dots,M, \quad t\ge 0. \tag{3}$$

Assume that $(A_j, \bar B_j)$ is stabilizable. Then there exist the controller gains $\bar K_{ij} \in \mathbb{R}^{k_j^u\times k_j}$ $(i=0,\dots,n-1)$ such that

$$D_j = A_j + \bar B_j \bar K_j, \quad \bar K_j = \left[\bar K_{0j},\dots,\bar K_{(n-1)j}\right] \tag{4}$$

is Hurwitz, i.e., subsystem (3) with small enough $r_j>0$ and $\|F_{lj}\|$, where $l\ne j$, is stabilized by the state-feedback

$$u_j(t) = \bar K_j x_j(t) = \sum_{i=0}^{n-1} \bar K_{ij} x_{ij}(t). \tag{5}$$

The closed-loop subsystem (3), (5) with $r_j=0$ takes the form

$$\dot x_j(t) = D_j x_j(t) + \sum_{l=1,l\ne j}^{M} F_{lj} x_l(t), \quad j=1,\dots,M, \quad t\ge 0 \tag{6}$$

with $D_j$ given by (4).

Note that when only the measurement $x_{0j}(t)=y_j(t)$ is available, similar to Fridman and Shaikhet (2016), Selivanov and Fridman (2018b) and Zhang and Fridman (2020) one can employ its current and past values via the finite-difference method to approximate the derivatives $x_{ij}(t)$ $(i=1,\dots,n-1)$ in (5):

$$\bar x_{0j}(t) = x_{0j}(t),$$

$$x_{ij}(t) \approx \bar x_{ij}(t) = \frac{\bar x_{(i-1)j}(t) - \bar x_{(i-1)j}(t-h_j)}{h_j}$$

$$= \frac{1}{h_j^i} \sum_{m=0}^{i} \binom{i}{m} (-1)^m \bar x_{0j}(t-mh_j), \tag{7}$$

where $h_j>0$ is a constant delay and $\binom{i}{m} = \frac{i!}{m!(i-m)!}$ is the binomial coefficient. Thus, if $x_{ij}(t)$ in (5) is replaced by $\bar x_{ij}(t)$ in (7), one can obtain

$$u_j(t) = \sum_{i=0}^{n-1} \bar K_{ij} \bar x_{ij}(t) = \sum_{i=0}^{n-1} K_{ij} x_{0j}(t-ih_j), \tag{8}$$

where $x_{0j}(t) = x_{0j}(0)$ for $t<0$ and

$$K_{ij} = (-1)^i \sum_{m=i}^{n-1} \binom{m}{i} \frac{1}{h_j^m} \bar K_{mj}, \quad i=0,\dots,n-1. \tag{9}$$

It should be pointed out that the feedback (8) with controller gains (9) is an ideal one since it depends on the accurate measurements $x_{0j}$. However, in the practical engineering e.g. power systems (Liu et al., 2020) and permanent magnet synchronous motor servo systems (Xie et al., 2018), the measurements are subject to unknown measurement disturbances, i.e.

$$\tilde x_{0j}(t) = x_{0j}(t) + \omega_j(t), \quad j=1,\dots,M, \tag{10}$$

where $\omega_j(t)$ is an unknown disturbance. As in Furtat et al. (2018), Sanz et al. (2016), we assume the following:

**Assumption 1.** The unknown disturbances $\omega_j(t)$ $(j = 1,\dots,M)$ are $(n-1)$-times continuously differentiable and are uniformly bounded together with their derivatives, i.e. $|\omega_j^{(i)}(t)| \le \bar{\omega}_{ij}$ $(i = 0,\dots,n-1)$ for all $t \ge 0$.

Our objective in this paper is to take into account measurement disturbances in the delayed implementation of the derivative-dependent controller to achieve ISS of the closed-loop subsystems. Considering the unknown disturbances, we present the following approximations:

$$\begin{aligned} x_{0j}(t) &\approx \hat{x}_{0j}(t) = \tilde{x}_{0j}(t),\\ x_{ij}(t) &\approx \hat{x}_{ij}(t) = \frac{\hat{x}_{(i-1)j}(t) - \hat{x}_{(i-1)j}(t-h_j)}{h_j}\\ &= \frac{1}{h_j^i}\sum_{m=0}^{i}\binom{i}{m}(-1)^m \tilde{x}_{0j}(t-mh_j), \end{aligned} \tag{11}$$

where $\tilde{x}_{0j}(t)$ is defined in (10). By replacing $x_{ij}(t)$ in (5) with $\hat{x}_{ij}(t)$ given by (11), we have the following delay-dependent feedback

$$u_j(t) = \sum_{i=0}^{n-1}\bar{K}_{ij}\hat{x}_{ij}(t) = \sum_{i=0}^{n-1}K_{ij}\tilde{x}_{0j}(t-ih_j), \tag{12}$$

where $\tilde{x}_{0j}(t) = \tilde{x}_{0j}(0)$ for $t<0$ and $K_{ij}$ $(i = 0,\dots,n-1)$ are given by (9).

**Remark 1.** Note that the advantage of the static output-feedback (12) which uses delayed measurements is its simplicity in the design and implementation compared to the observer-based design. However, these delayed feedbacks are robust with respect to small input delays (that are smaller than the feedback delays) (Fridman & Shaikhet, 2016) and to small and smooth disturbances. For larger input delays this design is not applicable.

Then the closed-loop subsystem (3), (12) has the following form

$$\begin{aligned} \dot{x}_j(t) &= A_j x_j(t) + \sum_{i=0}^{n-1}\bar{B}_j\bar{K}_{ij}\hat{x}_{ij}(t-r_j) + \sum_{l=1,l\ne j}^{M}F_{lj}x_l(t),\\ &j = 1,\dots,M, \quad t \ge 0. \end{aligned} \tag{13}$$

Inspired by Selivanov and Fridman (2018b), we present now a transformation without any approximations for system (13). Using (7), (10) and (11), we first present the terms $\hat{x}_{ij}(t-r_j)$ $(i = 0,\dots,n-1)$ as

$$\begin{aligned} \hat{x}_{0j}(t-r_j) &= x_{0j}(t-r_j) + v_{0j}(t-r_j)\\ &= x_{0j}(t) - \int_{t-r_j}^{t} x_{1j}(s)ds + v_{0j}(t-r_j),\\ \hat{x}_{ij}(t-r_j) &= \bar{x}_{ij}(t-r_j) + v_{ij}(t-r_j)\\ &= \bar{x}_{ij}(t) - \int_{t-r_j}^{t}\dot{\bar{x}}_{ij}(s)ds + v_{ij}(t-r_j), \quad i = 1,\dots,n-1, \end{aligned} \tag{14}$$

where $v_{0j}(t) = \omega_j(t)$ and

$$v_{ij}(t) = \frac{1}{h_j^i}\int_{t-h_j}^{t}\underbrace{\int_{s_1-h_j}^{s_1}\cdots\int_{s_{i-1}-h_j}^{s_{i-1}}}_{i-1}\omega_j^{(i)}(s_i)ds_i\dots ds_1, \quad i = 1,\dots,n-1. \tag{15}$$

Under Assumption 1, we have

$$|v_{ij}(t)| \le \bar{\omega}_{ij}, \quad t \ge (n-1)h_j, \quad i = 0,\dots,n-1, \quad j = 1,\dots,M. \tag{16}$$

Moreover, the error between $\bar{x}_{ij}(t)$ and $x_{ij}(t)$ is expressed by Selivanov and Fridman (2018b)

$$\bar{x}_{ij}(t) = x_{ij}(t) - \int_{t-ih_j}^{t}\varphi_{ij}(t-s)\dot{x}_{ij}(s)ds, \quad i = 1,\dots,n-1, \tag{17}$$

where $\varphi_{1j}(v) = \frac{h_j - v}{h_j}$, $v \in [0,h_j]$ and for $i = 1,\dots,n-2$

$$\varphi_{(i+1)j}(v) = \begin{cases} \int_0^v \frac{\varphi_{ij}(\lambda)}{h_j}d\lambda + \frac{h_j-v}{h_j}, & v\in[0,h_j],\\ \int_{v-h_j}^{v}\frac{\varphi_{ij}(\lambda)}{h_j}d\lambda, & v\in(h_j, ih_j),\\ \int_{v-h_j}^{ih_j}\frac{\varphi_{ij}(\lambda)}{h_j}d\lambda, & v\in[ih_j, ih_j+h_j]. \end{cases} \tag{18}$$

Let

$$\begin{aligned} &\phi_{ij}(\lambda) = \int_\lambda^{ih_j}\varphi_{ij}(v)dv, \quad \psi_{ij}(v) = -\frac{d}{dv}\varphi_{ij}(v), \quad v\in[0,ih_j],\\ &i = 1,\dots,n-1. \end{aligned} \tag{19}$$

The functions $\varphi_{ij}(v)$ $(i = 1,\dots,n-1)$ have the following properties (Selivanov & Fridman, 2018b; Zhang & Fridman, 2020)

$$\begin{aligned} &0 \le \varphi_{ij}(v) \le 1, \quad \varphi_{ij}(0) = 1, \quad \varphi_{ij}(ih_j) = 0,\\ &\phi_{ij}(0) = \frac{ih_j}{2}, \quad \psi_{ij}(v) \in [0, \frac{1}{h_j}], \quad v\in[0,ih_j]. \end{aligned} \tag{20}$$

From (17) and (20) it follows that

$$\dot{\bar{x}}_{ij}(t) = \int_{t-ih_j}^{t}\psi_{ij}(t-s)\dot{x}_{ij}(s)ds, \quad i = 1,\dots,n-1. \tag{21}$$

Based on (17) and (21), we have

$$\begin{aligned} \bar{x}_{ij}(t-r_j) &= x_{ij}(t) - \int_{t-ih_j}^{t}\varphi_{ij}(t-s)\dot{x}_{ij}(s)ds\\ &- \int_{t-r_j}^{t}\int_{s-ih_j}^{s}\psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds, \quad i = 1,\dots,n-1. \end{aligned} \tag{22}$$

Finally, denoting

$$\begin{aligned} &\rho_{ij}(t) = -\int_{t-ih_j}^{t}\varphi_{ij}(t-s)\dot{x}_{ij}(s)ds, \quad \kappa_{0j}(t) = -\int_{t-r_j}^{t}x_{1j}(s)ds,\\ &\kappa_{ij}(t) = -\int_{t-r_j}^{t}\int_{s-ih_j}^{s}\psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds, \quad i = 1,\dots,n-1 \end{aligned} \tag{23}$$

and employing (14), system (13) can be rewritten as

$$\begin{aligned} \dot{x}_j(t) &= D_j x_j(t) + \bar{B}_j\bar{K}_{0j}(\kappa_{0j}(t) + v_{0j}(t-r_j))\\ &+ \sum_{i=1}^{n-1}\bar{B}_j\bar{K}_{ij}(\rho_{ij}(t) + \kappa_{ij}(t) + v_{ij}(t-r_j)) + \sum_{l=1,l\ne j}^{M}F_{lj}x_l(t), \end{aligned} \tag{24}$$

where $D_j$ is defined by (4). Clearly, compared to system (6), system (24) involves the additional errors $\rho_{ij}$, $\kappa_{ij}$ (that are dependent of $h_j$ or $r_j$) and the disturbances $v_{ij}$ due to the approximation (11) via the finite difference method. If $h_j$ and $r_j$ grow, these errors will ruin the system stability. To handle these errors, in the stability analysis we add appropriate terms to the corresponding LKFs. Thus, we choose the following Lyapunov functional for $t \ge (n-1)h_j + r_j$:

$$V_j(t) = V_{0j}(t) + V_{\kappa_{0j}}(t) + \sum_{i=1}^{n-1}\left(V_{\rho_{ij}}(t) + V_{\kappa_{ij}}(t) + \tilde{V}_{\kappa_{ij}}(t)\right), \tag{25}$$

where

$$\begin{aligned} V_{0j}(t) &= x_j^T(t)P_j x_j(t),\\ V_{\rho_{ij}}(t) &= 2ih_j\int_{t-ih_j}^{t}e^{-2\alpha_j(t-s)}\phi_{ij}(t-s)\dot{x}_{ij}^T(s)R_{ij}\dot{x}_{ij}(s)ds,\\ V_{\kappa_{0j}}(t) &= r_j\int_{t-r_j}^{t}e^{-2\alpha_j(t-s)}(s-t+r_j)x_{1j}^T(s)Q_{0j}x_{1j}(s)ds,\\ V_{\kappa_{ij}}(t) &= \frac{ir_j}{h_j}\int_{t-r_j}^{t}\int_{s-ih_j}^{s}e^{-2\alpha_j(t-s)}(s-t+r_j)\dot{x}_{ij}^T(\theta)Q_{ij}\dot{x}_{ij}(\theta)d\theta ds,\\ \tilde{V}_{\kappa_{ij}}(t) &= \frac{ir_j^2}{h_j}\int_{t-ih_j}^{t}e^{-2\alpha_j(t-\theta-ih_j)}(\theta-t+ih_j)\dot{x}_{ij}^T(\theta)Q_{ij}\dot{x}_{ij}(\theta)d\theta \end{aligned} \tag{26}$$

with $nk_j \times nk_j$ matrix $P>0$ and $k_j \times k_j > 0$ matrices $Q_{0j}>0$, $R_{ij}>0$ and $Q_{ij}>0$, $i = 1,\dots,n-1$. Note that the terms $V_{\rho_{ij}}(t)$ $(i = 1,\dots,n-1)$

borrowed from Selivanov and Fridman (2018b) compensate $\rho_{ij}(t)$, the terms $V_{\kappa_{ij}}(t)$ $(i = 0, \dots, n-1)$ compensate $\kappa_{ij}(t)$ whereas the terms $\bar{V}_{\kappa_{ij}}(t)$ $(i = 1, \dots, n-1)$ are suggested to cancel the positive term from $\dot{V}_{\kappa_{ij}}(t)$.

Based on the Lyapunov functional $V_j(t)$ defined in (25), we derive the following sufficient delay-dependent conditions that allow to find the upper bounds on $h_j$ and $r_j$ ensuring the stability.

**Theorem 1.** *Let Assumption 1 hold. Given gains $\bar{K}_{ij}$ $(i = 0, \dots, n-1)$, let the derivative-dependent feedback (5) exponentially stabilize subsystem (3), where $r_j = 0$ and $F_{lj} = 0$ $(l \neq j)$, with a decay rate $\bar{\alpha}_j > 0$.*

*(i) Given tuning parameters $\delta > 0$, $r_j > 0$, $h_j > 0$, $\alpha_j > 0$ and $\beta_{lj} > 0$ $(j, l = 1, \dots, M,\ l \neq j)$ such that the following Metzler matrix*

$$\mathcal{M} = \begin{bmatrix} -\alpha_1 & \beta_{21} & \dots & \beta_{M1} \\ \beta_{12} & -\alpha_2 & \dots & \beta_{M2} \\ \dots & \dots & \dots & \dots \\ \beta_{1M} & \beta_{2M} & \dots & -\alpha_M \end{bmatrix} \tag{27}$$

*is Hurwitz, let there exist $nk_l \times nk_l$ matrix $P_l > 0$ $(l = 1, \dots, M)$, $k_j \times k_j$ matrices $Q_{0j} > 0$, $R_{ij} > 0$, $Q_{ij} > 0$ $(i = 1, \dots, n-1)$, $M \times M$ matrix $\mathcal{P} > 0$, and scalars $\gamma_{ij} > 0$ $(i = 0, \dots, n-1)$, $\Gamma > 0$ that satisfy*

$$\begin{bmatrix} 2\mathcal{P}\mathcal{M} + 2\mathcal{M}^T\mathcal{P} + 4\delta\mathcal{P} & \mathcal{P} \\ * & -\Gamma^2 I \end{bmatrix} \leq 0, \tag{28}$$

$$\Phi_j < 0, \quad j = 1, \dots, M, \tag{29}$$

*where $\Phi_j$ is the symmetric matrix composed of*

$$\begin{aligned}
\Phi_{11}^j &= P_j D_j + D_j^T P_j + 2\alpha_j P_j + r_j^2 H_1^T Q_{0j} H_1 + \sum_{i=1}^{n-2} i^2 H_{i+1}^T \Big( h_j^2 R_{ij} \\
&\quad + r_j^2 e^{2\alpha_j i h_j} Q_{ij} \Big) H_{i+1}, \quad \Phi_{12}^j = \Phi_{14}^j = P_j \bar{B}_j [\bar{K}_{1j}, \dots, \bar{K}_{(n-1)j}], \\
\Phi_{13}^j &= P_j \bar{B}_j \bar{K}_{0j}, \quad \Phi_{15}^j = \mathrm{row}_{l=1,\dots,M, l\neq j}\{P_j F_{lj}\}, \quad \Phi_{17}^j = D_j^T H_{n-1}^T \Lambda_j, \\
\Phi_{16}^j &= P_j \bar{B}_j [\bar{K}_{0j}, \dots, \bar{K}_{(n-1)j}], \qquad \Phi_{22}^j = -\mathrm{diag}\{4e^{-2\alpha_j i h_j} R_{ij}\}_{i=1}^{n-1}, \\
\Phi_{27}^j &= \Phi_{47}^j = [\bar{K}_{1j}, \dots, \bar{K}_{(n-1)j}]^T \bar{B}_j^T H_{n-1}^T \Lambda_j, \quad \Phi_{33}^j = -e^{-2\alpha_j r_j} Q_{0j}, \\
\Phi_{37}^j &= \bar{K}_{0j}^T \bar{B}_j^T H_{n-1}^T \Lambda_j, \quad \Phi_{44}^j = -\mathrm{diag}\{e^{-2\alpha_j r_j} Q_{ij}\}_{i=1}^{n-1}, \quad \Phi_{77}^j = -\Lambda_j, \\
\Phi_{55}^j &= -\mathrm{diag}_{l=1,\dots,M, l\neq j}\{2\beta_{lj} P_l\}, \quad \Phi_{57}^j = \mathrm{row}_{l=1,\dots,M, l\neq j}\{F_{lj}^T H_{n-1}^T \Lambda_j\} \\
\Phi_{67}^j &= [\bar{K}_{0j}, \dots, \bar{K}_{(n-1)j}]^T \bar{B}_j^T H_{n-1}^T \Lambda_j, \quad \Phi_{66}^j = -\mathrm{diag}\{\gamma_{ij}^2 I\}_{i=0}^{n-1}
\end{aligned} \tag{30}$$

*and other blocks are zero matrices. Here $D_j$ is given by (4) and*

$$\begin{aligned}
\Lambda_j &= (n-1)^2 \left( h_j^2 R_{(n-1)j} + r_j^2 e^{2\alpha_j (n-1) h_j} Q_{(n-1)j} \right), \\
H_i &= [0_{k\times ik}, I_k, 0_{k\times(n-i-1)k}], \quad i = 1, \dots, n-1.
\end{aligned} \tag{31}$$

*Then solution of subsystem (3) under the delay-dependent feedback (12) with controller gains (9) satisfies for $t \geq (n-1)\bar{h} + \bar{r}$*

$$\begin{aligned}
\lambda_{\min}^2\{P_j\}|x_j(t)|^4 &\leq e^{-4\delta(t-(n-1)\bar{h}-\bar{r})} \frac{\lambda_{\max}\{\mathcal{P}\}}{\lambda_{\min}\{\mathcal{P}\}} |z((n-1)\bar{h}+\bar{r})|^2 \\
&\quad + (1 - e^{-4\delta(t-(n-1)\bar{h}-\bar{r})}) \frac{\Gamma^2}{4\delta\lambda_{\min}\{\mathcal{P}\}} \\
&\quad \times \sum_{j=1}^{M} (\sum_{i=0}^{n-1} \gamma_{ij}^2 \|v_{ij}[(n-1)\bar{h} + \bar{r} - r_j, t - r_j]\|_\infty^2)^2,
\end{aligned} \tag{32}$$

*where $z((n-1)\bar{h}+\bar{r}) \geq\geq V((n-1)\bar{h}+\bar{r}) = [V_1((n-1)\bar{h}+\bar{r}), \dots, V_M(n-1)\bar{h}+\bar{r}]^T$ with $V_j$ defined in (25), $v_{0j}(t) = \omega_j(t)$ and $v_{ij}$ $(j = 1, \dots, n-1)$ are given by (15), and*

$$\bar{h} = \max_{j=1,\dots,M}\{h_j\}, \quad \bar{r} = \max_{j=1,\dots,M}\{r_j\}. \tag{33}$$

*Moreover, from (16), given $\Delta = \sum_{j=1}^M \left(\sum_{i=0}^{n-1} \gamma_{ij}^2 \bar{\omega}_{ij}^2\right)^2$ the ellipsoid*

$$\chi_\infty = \{x_j \in \mathbb{R}^{nk_j} : \lambda_{\min}\{P_j\}|x_j(t)|^2 \leq \frac{\Gamma\sqrt{\Delta}}{2\sqrt{\delta\lambda_{\min}\{\mathcal{P}\}}}\} \tag{34}$$

*is exponentially attractive with a delay rate $\delta$ for all $x_j(t_0) \in \mathbb{R}^{nk_j}$ and $(n-1)$-times continuously differentiable and uniformly bounded $\omega_j(t)$ together with their derivatives, i.e. $|\omega_j^{(i)}(t)| \leq \bar{\omega}_{ij}$ $(i = 0, \dots, n-1)$ for all $t \geq 0$.*

*(ii) Given any $\alpha_j \in (0, \bar{\alpha}_j)$ and $\beta_{lj} > 0$ $(l = 1, \dots, M,\ l \neq j)$ such that the Metzler matrix $\mathcal{M}$ defined by (27) is Hurwitz, LMIs of item (i) are always feasible for small enough $r_j > 0$, $h_j > 0$, $\|F_{lj}\|$ $(l \neq j)$, $\gamma_{ij}^{-1} > 0$ $(i = 1, \dots, n-1)$, $\Gamma^{-1} > 0$ and $\delta > 0$.*

**Proof.** (i) Differentiating $V_j(t)$ given by (25) along the closed-loop subsystem (24), we have

$$\begin{aligned}
\dot{V}_{0j}(t) &= 2x_j^T(t) P_j [D_j x_j(t) + \bar{B}_j \bar{K}_{0j}(\kappa_{0j}(t) + v_{0j}(t - r_j)) \\
&\quad + \sum_{i=1}^{n-1} \bar{B}_j \bar{K}_{ij}(\rho_{ij}(t) + \kappa_{ij}(t) + v_{ij}(t - r_j)) + \sum_{l=1, l\neq j}^{M} F_{lj} x_l(t)],
\end{aligned} \tag{35}$$

$$\begin{aligned}
\dot{V}_{\rho_{ij}}(t) + 2\alpha_j V_{\rho_{ij}}(t) &= (ih_j)^2 \dot{x}_{ij}^T(t) R_{ij} \dot{x}_{ij}(t) \\
&\quad - 2ih_j \int_{t-ih_j}^{t} e^{-2\alpha_j(t-s)} \varphi_{ij}(t-s) \dot{x}_{ij}^T(s) R_{ij} \dot{x}_{ij}(s) ds,
\end{aligned} \tag{36}$$

$$\begin{aligned}
\dot{V}_{\kappa_{0j}}(t) + 2\alpha_j V_{\kappa_{0j}}(t) &= r_j^2 x_{1j}^T(t) Q_{0j} x_{1j}(t) \\
&\quad - r_j \int_{t-r_j}^{t} e^{-2\alpha_j(t-s)} x_{1j}^T(s) Q_{0j} x_{1j}(s) ds,
\end{aligned} \tag{37}$$

$$\begin{aligned}
\dot{V}_{\kappa_{ij}}(t) + 2\alpha_j V_{\kappa_{ij}}(t) &= \frac{ir_j^2}{h_j} \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta \\
&\quad - \frac{ir_j}{h_j} \int_{t-r_j}^{t} \int_{s-ih_j}^{s} e^{-2\alpha_j(t-s)} \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta ds \\
&\leq \frac{ir_j^2}{h_j} \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta \\
&\quad - ih_j r_j \int_{t-r_j}^{t} \int_{s-ih_j}^{s} \psi_{ij}^2(s-\theta) \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta ds,
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
\dot{\bar{V}}_{\kappa_{ij}}(t) + 2\alpha_j \bar{V}_{\kappa_{ij}}(t) &= (ir_j)^2 e^{2\alpha_j i h_j} \dot{x}_{ij}^T(t) Q_{ij} \dot{x}_{ij}(t) \\
&\quad - \frac{ir_j^2}{h_j} \int_{t-ih_j}^{t} e^{-2\alpha_j(t-\theta-ih_j)} \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta \\
&\leq (ir_j)^2 e^{2\alpha_j i h_j} \dot{x}_{ij}^T(t) Q_{ij} \dot{x}_{ij}(t) - \frac{ir_j^2}{h_j} \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta.
\end{aligned} \tag{39}$$

Based on Jensen's inequality (see (3.87) in Fridman (2014)) and its extended version (Lemma 1), we have

$$2ih_j \int_{t-ih_j}^{t} \varphi_{ij}(t-s) \dot{x}_{ij}^T(s) R_{ij} \dot{x}_{ij}(s) ds \geq 4\rho_{ij}^T(t) R_{ij} \rho_{ij}(t), \tag{40}$$

$$r_j \int_{t-r_j}^{t} x_{1j}^T(s) Q_{0j} x_{1j}(s) ds \geq \kappa_{0j}^T(t) Q_{0j} \kappa_{0j}(t), \tag{41}$$

and

$$ih_j r_j \int_{t-r_j}^{t} \int_{s-ih_j}^{s} \psi_{ij}^2(s-\theta) \dot{x}_{ij}^T(\theta) Q_{ij} \dot{x}_{ij}(\theta) d\theta ds \geq \kappa_{ij}^T(t) Q_{ij} \kappa_{ij}(t). \tag{42}$$

From (35)–(42), we find

$$\begin{aligned}
&\dot{V}_j(t) + 2\alpha_j V_j(t) - 2\sum_{l=1, l\neq j}^{M} \beta_{lj} x_l^T(t) P_l x_l(t) - \sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t - r_j)|^2 \\
&\leq \zeta_j^T(t) \tilde{\Phi}_j \zeta_j(t) + \dot{x}_{(n-1)j}^T(t) \Lambda_j \dot{x}_{(n-1)j}(t),
\end{aligned} \tag{43}$$

where $\tilde{\Phi}_j$ is obtained from $\Phi_j$ (composed of (30)) by taking away the last block-column and block-row, $\Lambda_j$ is given by (31), and

$$\begin{aligned}
&\zeta_j(t) = \mathrm{col}\{\bar{\zeta}_j(t), \bar{v}_j(t)\}, \quad \bar{\zeta}_j(t) = \mathrm{col}\{x_j(t), \bar{\rho}_j(t), \bar{\kappa}_j(t), X_j(t)\} \\
&\bar{\rho}_j(t) = \mathrm{col}_{i=1,\dots,n-1}\{\rho_{ij}(t)\}, \quad \bar{\kappa}_j(t) = \mathrm{col}_{i=0,\dots,n-1}\{\kappa_{ij}(t)\},
\end{aligned}$$

$$X_j(t) = \mathrm{col}_{l=1,\dots,M,l\neq j}\{x_l(t)\}, \quad \bar{v}_j(t) = \mathrm{col}_{i=0,\dots,n-1}\{v_{ij}(t-r_j)\}. \tag{44}$$

Substituting $\dot{x}_{(n-1)j}(t) = H_{n-1}\dot{x}_j(t)$ into (43), where $\dot{x}_j(t)$ and $H_{n-1}$ are, respectively, defined in (24) and (31) and then applying Schur complement lead to

$$\begin{aligned}
\dot{V}_j(t) + 2\alpha_j V_j(t) &\le \sum_{l=1,l\neq j}^{M} 2\beta_{lj} x_l^T(t) P_l x_l(t) \\
&+ \sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2 \\
&\le \sum_{l=1,l\neq j}^{M} 2\beta_{lj} V_l(t) + \sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2, \quad t \ge (n-1)h_j + r_j.
\end{aligned}$$

Then the following holds for all $j = 1,\dots,M$ and $t \ge (n-1)\bar{h} + \bar{r}$

$$\dot{V}_j(t) \le -2\alpha_j V_j(t) + \sum_{l=1,l\neq j}^{M} 2\beta_{lj} V_l(t) + \sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2, \tag{45}$$

where $\bar{h}$ and $\bar{r}$ are defined in (33).

Define a vector LKF $V(t) = \mathrm{col}\{V_1(t),\dots,V_M(t)\}$. From (45), it follows that

$$\dot{V}(t) \le 2\mathcal{M}V(t) + \mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2\}, \quad t \ge (n-1)\bar{h} + \bar{r}$$

with Metzler and Hurwitz $\mathcal{M}$ defined by (27). Following arguments of Theorem 3.2 in Nersesov and Haddad (2006), we find that given the same $\mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2\}$ and any initial conditions $z((n-1)\bar{h}+\bar{r}) \in \mathbb{R}^M$ satisfying $z((n-1)\bar{h}+\bar{r}) \geq\geq V((n-1)\bar{h}+\bar{r})$, we have $V(t) \leq\leq z(t)$ for $t \ge (n-1)\bar{h}+\bar{r}$, where $z(t)$ is a solution to $\dot{z}(t) = 2\mathcal{M}z(t) + \mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2\}$. Thus, $V_j(t) \le z_j(t)$ for $j = 1,\dots,M$, where $z_j$ is the $j$th component of $z$. Since $z_j(t) \le |z(t)|$ holds for all $j = 1,\dots,M$, we obtain

$$V_j(t) \le |z(t)|, \quad j = 1,\dots,M, \quad t \ge (n-1)\bar{h} + \bar{r}. \tag{46}$$

We next present the upper bound on $|z(t)|$, where $z(t)$ is a solution to $\dot{z}(t) = 2\mathcal{M}z(t) + \mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2\}$. Define a Lyapunov function $\mathcal{V}(t) = z(t)^T \mathcal{P} z(t)$, where $\mathcal{P} > 0$ and $t \ge (n-1)\bar{h}+\bar{r}$. Then we have for $t \ge (n-1)\bar{h}+\bar{r}$

$$\begin{aligned}
&\dot{\mathcal{V}}(t) + 4\delta\mathcal{V}(t) - \Gamma^2 \left|\mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2\}\right|^2 \\
&= \varsigma^T(t) \begin{bmatrix} 2\mathcal{P}\mathcal{M} + 2\mathcal{M}^T\mathcal{P} + 4\delta\mathcal{P} & \mathcal{P} \\ * & -\Gamma^2 I \end{bmatrix} \varsigma(t) \le 0,
\end{aligned}$$

where

$$\varsigma(t) = \begin{bmatrix} z(t) \\ \mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2\} \end{bmatrix}$$

Thus

$$\begin{aligned}
\mathcal{V}(t) &\le e^{-4\delta(t-(n-1)\bar{h}-\bar{r})}\mathcal{V}((n-1)\bar{h}+\bar{r}) \\
&+ \int_{(n-1)\bar{h}+\bar{r}}^{t} e^{-4\delta(t-s)}\Gamma^2 \left|\mathrm{col}_{j=1,\dots,M}\{\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(s-r_j)|^2\}\right|^2 ds \\
&\le e^{-4\delta(t-(n-1)\bar{h}-\bar{r})}\mathcal{V}((n-1)\bar{h}+\bar{r}) + (1 - e^{-4\delta(t-(n-1)\bar{h}-\bar{r})})\frac{\Gamma^2}{4\delta} \\
&\times \sum_{j=1}^{M}(\sum_{i=0}^{n-1} \gamma_{ij}^2 \|v_{ij}[(n-1)\bar{h}+\bar{r}-r_j, t-r_j]\|_\infty^2)^2.
\end{aligned}$$

Since $\lambda_{\min}\{\mathcal{P}\}|z(t)|^2 \le \mathcal{V}(t) \le \lambda_{\max}\{\mathcal{P}\}|z(t)|^2$, we have for all $t \ge (n-1)\bar{h}+\bar{r}$

$$\begin{aligned}
&\lambda_{\min}\{\mathcal{P}\}|z(t)|^2 \le e^{-4\delta(t-(n-1)\bar{h}-\bar{r})}\lambda_{\max}\{\mathcal{P}\}|z((n-1)\bar{h}+\bar{r})|^2 \\
&+ (1 - e^{-4\delta(t-(n-1)\bar{h}-\bar{r})})\frac{\Gamma^2}{4\delta}\sum_{j=1}^{M}(\sum_{i=0}^{n-1} \gamma_{ij}^2 \|v_{ij}[(n-1)\bar{h}+\bar{r}-r_j, t-r_j]\|_\infty^2)^2.
\end{aligned} \tag{47}$$

Taking into account $\lambda_{\min}\{P_j\}|x_j(t)|^2 \le x_j^T P_j x_j \le V_j(t) \le |z(t)|$, we find that solution of the closed-loop subsystem (1), (12) satisfies (32).

(ii) If (5) exponentially stabilizes (3), where $r_j = 0$ and $F_{lj} = 0$ $(l \neq j)$, with a decay rate $\bar{\alpha}_j > 0$, then for any $\alpha_j \in (0, \bar{\alpha}_j)$ there exists $0 < P_j \in \mathbb{R}^{nk_j \times nk_j}$ such that

$$P_j D_j + D_j^T P_j + 2\alpha_j P_j < 0. \tag{48}$$

We choose $R_{ij} = \frac{1}{h_j}I$, $Q_{0j} = Q_{ij} = \frac{1}{r_j}I$ and $F_{lj} = \beta_{lj}I$ $(l \neq j)$. By Schur complement, $\tilde{\Phi}_j < 0$ is equivalent to

$$\begin{aligned}
&P_j D_j + D_j^T P_j + 2\alpha_j P_j + O(r_j I) + O(h_j I) \\
&+ O(\gamma_{ij}^{-2} I) + \sum_{l=1,l\neq j}^{M} \frac{\beta_{lj}}{2} P_j^T P_l^{-1} P_j < 0.
\end{aligned} \tag{49}$$

Inequality (48) implies (49) for small enough $r_j > 0$, $h_j > 0$, $\beta_{lj} > 0$ (i.e. $\|F_{lj}\|$) and $\gamma_{ij}^{-1} > 0$ $(i = 0,\dots,n-1)$ since $O(r_j I) + O(h_j I) + O(\gamma_{ij}^{-2} I) + \sum_{l=1,l\neq j}^{M} \frac{\beta_{lj}}{2} P_j^T P_l^{-1} P_j \to 0$ for $r_j \to 0$, $h_j \to 0$, $\beta_{lj} \to 0$ and $\gamma_{ij}^{-1} \to 0$ $(i = 0,\dots,n-1)$. Then, applying Schur complement to the last block-column and block-row of $\Phi_j$ composed of (30), we find that $\Phi_j < 0$ for small enough $r_j > 0$ and $h_j > 0$ if $\tilde{\Phi}_j < 0$ is feasible. Moreover, the second inequality of (29) is always feasible for small enough $\delta > 0$ and $\Gamma^{-1} > 0$. Therefore, LMIs of item (i) are always feasible for small enough $r_j > 0$, $h_j > 0$, $\|F_{lj}\|$ $(l \neq j)$, $\gamma_{ij}^{-1} > 0$ $(i = 0,\dots,n-1)$, $\Gamma^{-1} > 0$ and $\delta > 0$. This completes the proof. □

**Remark 2.** Note that to deal with the coupling term $\sum_{l=1,l\neq j}^{M} F_{lj} x_l(t)$, in stability analysis we subtract the term $2\sum_{l=1,l\neq j}^{M} \beta_{lj} x_l^T(t) P_l x_l(t)$ from $\dot{V}_j(t) + 2\alpha_j V_j(t)$, see (43). Moreover, in the present paper we suggested simple Lyapunov functionals leading to efficient results and we expect that in the future the results may be improved e.g. by using advanced Lyapunov–based methods (for example, by using augmented Lyapunov functionals with appropriate integral inequalities (Fridman, 2014; Seuret & Gouaisbaut, 2013)).

**Remark 3.** If one applies the existing method (Freirich & Fridman, 2016; Zhu & Fridman, 2020b), i.e. defining $\bar{V}(t) = \sum_{j=1}^{M} V_j(t)$, from (45) we obtain

$$\begin{aligned}
\dot{\bar{V}}(t) &\le -\sum_{j=1}^{M} 2(\alpha_j - \sum_{l=1,l\neq j}^{M} \beta_{jl}) V_j(t) + \sum_{j=1}^{M}\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2 \\
&\le -2\bar{\delta}\bar{V}(t) + \sum_{j=1}^{M}\sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(t-r_j)|^2,
\end{aligned}$$

where $\bar{\delta} = \min_{j=1,\dots,M}\left\{\alpha_j - \sum_{l=1,l\neq j}^{M} \beta_{jl}\right\}$. Then if $\Phi_j < 0$ in (29) holds, the closed-loop system is ISS provided $\bar{\delta} > 0$, i.e. $\alpha_j - \sum_{l=1,l\neq j}^{M} \beta_{jl} > 0$ for $j = 1,\dots,M$. Thus, the latter restrictive condition is avoided in this paper since we introduced the Metzler and Hurwitz $\mathcal{M}$ defined by (27). Using (27), one may choose larger $\beta_{jl}$ that allows larger coupling than the existing method (Freirich & Fridman, 2016; Zhu & Fridman, 2020b) (see Example 2 below).

**Remark 4.** To select the tuning parameters of matrix $\mathcal{M}$ given by (27) as well as $h_j$ and $r_j$, we suggest the following algorithm: choose $\bar{K}_j$ via pole-placement such that the state-feedback (5) exponentially stabilizes (3), where $r_j = 0$ and $F_{lj} = 0$ $(l \neq j)$, with a decay rate $\bar{\alpha}_j > 0$. By solving LMIs $\Phi_j < 0$ with small enough $h_j > 0$, $r_j > 0$ and $\beta_{lj}^{-1}$ $(l \neq j)$, we find critical maximal values of $\alpha_j$ as $\alpha_j^* < \bar{\alpha}_j$. Next, by choosing $\alpha_j \in (0, \alpha_j^*]$, we decrease each $\beta_{lj}$ until that matrix $\mathcal{M}$ is Hurwitz while ensuring the feasibility of LMIs $\Phi_j < 0$. Thus, we can obtain critical maximal values of $\beta_{lj}$ as $\beta_{lj}^*$. For $\beta_{lj}$ that are slightly smaller than $\beta_{lj}^*$, we can obtain critical maximal values of $h_j$ and $r_j$ as $h_j^*$ and $r_j^*$, respectively, such that for $h_j > h_j^*$ and $r_j > r_j^*$ LMIs $\Phi_j < 0$ become unfeasible.

## 3. Sampled-data delayed control

In this section, we consider sampled-data implementation for the delay-dependent feedback (12), which is more practical. We assume that the noisy measurements $\tilde{x}_{0j}(s_k^j)$ are available only at discrete sampling instants $s_k^j = kh_j$, where $h_j > 0$ is a sampling period and $k \in \mathbb{N}_0$. The derivative-dependent controller (12) is approximated by the sampled-data delayed controller

$$u_j(t) = \sum_{i=0}^{n-1} \bar{K}_{ij}\hat{x}_{ij}(s_k^j) = \sum_{i=0}^{n-1} K_{ij}\tilde{x}_{0j}(s_{k-i}^j), \quad t \in [s_k^j, s_{k+1}^j), \quad k \in \mathbb{N}_0, \tag{50}$$

where $x_{0j}(t) = x_{0j}(0)$ for $t < 0$ and $K_{ij}$ is given by (9). Note that the feedback (50) depends only on $n$ discrete-time measurements $\tilde{x}_{0j}(s_{k-n+1}^j), \dots, \tilde{x}_{0j}(s_k^j)$, which is easy to implement. One may store in the buffer $n-1$ measurements $\tilde{x}_{0j}(s_{k-n+1}^j), \dots, \tilde{x}_{0j}(s_{k-1}^j)$.

Assume also that Assumption 1 holds. Considering the input delay $r_j$, the sampled-data delayed controller (50) can be rewritten as

$$u_j(t-r_j) = \sum_{i=0}^{n-1} \bar{K}_{ij}\hat{x}_{ij}(s_k^j) = \sum_{i=0}^{n-1} K_{ij}\tilde{x}_{0j}(s_{k-i}^j), \quad t \in [s_k^j + r_j, s_{k+1}^j + r_j). \tag{51}$$

For $t \in [s_k^j + r_j, s_{k+1}^j + r_j)$ with $k \geq n-1$, we present the sampled measurements as

$$\begin{aligned}
\hat{x}_{0j}(s_k^j) &= x_{0j}(s_k^j) + v_{0j}(s_k^j) \\
&= x_{0j}(t) - \int_{s_k^j}^t x_{1j}(s)ds + v_{0j}(s_k^j), \\
\hat{x}_{ij}(s_k^j) &= \bar{x}_{ij}(s_k^j) + v_{ij}(s_k^j) \\
&= \bar{x}_{ij}(t) - \int_{s_k^j}^t \dot{\bar{x}}_{ij}(s)ds + v_{ij}(s_k^j) \\
&= x_{ij}(t) - \int_{t-ih_j}^t \varphi_{ij}(t-s)\dot{x}_{ij}(s)ds + v_{ij}(s_k^j) \\
&\quad - \int_{s_k^j}^t \int_{s-ih_j}^s \psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds, \quad i = 1, \dots, n-1.
\end{aligned} \tag{52}$$

Then using (51) and (52), the closed-loop system (3), (50) takes the form

$$\begin{aligned}
\dot{x}_j(t) &= D_j x_j(t) + \bar{B}_j\bar{K}_{0j}(\delta_{0j}(t) + v_{0j}(s_k^j)) + \sum_{i=1}^{n-1}\bar{B}_j\bar{K}_{ij}(\rho_{ij}(t) \\
&\quad + \delta_{ij}(t) + v_{ij}(s_k^j)) + \sum_{l=1,l\neq j}^{M} F_{lj}x_l(t), \quad t \in [s_k^j + r_j, s_{k+1}^j + r_j),
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
\delta_{0j}(t) &= -\int_{s_k^j}^t x_{1j}(s)ds, \\
\delta_{ij}(t) &= -\int_{s_k^j}^t \int_{s-ih_j}^s \psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds, \quad i = 1, \dots, n-1
\end{aligned} \tag{54}$$

with $D_j$ and $\rho_{ij}(t)$ given by (4) and (23), respectively.

**Theorem 2.** *Let Assumption 1 hold. Given gains $\bar{K}_{ij}$ $(i = 0, \dots, n-1)$, let the derivative-dependent feedback (5) exponentially stabilize subsystem (3), where $r_j = 0$ and $F_{lj} = 0$ $(l \neq j)$, with a decay rate $\bar{\alpha}_j > 0$.*

*(i) Given tuning parameters $\delta > 0$, $r_j > 0$, $h_j > 0$, $\alpha_j > 0$ and $\beta_{lj} > 0$ $(j, l = 1, \dots, M,\ l \neq j)$ such that the Metzler matrix $\mathcal{M}$ defined by (27) is Hurwitz, let there exist $nk_l \times nk_l$ matrix $P_l > 0$ $(l = 1, \dots, M)$, $k_j \times k_j$ matrices $R_{ij} > 0$, $Q_{ij} > 0$, $W_{ij} > 0$ $(i = 0, \dots, n-1)$, $M \times M$ matrix $\mathcal{P} > 0$, and scalars $\gamma_{ij} > 0$ $(i = 0, \dots, n-1)$, $\Gamma > 0$ that satisfy (28) and*

$$\Xi_j < 0, \quad j = 1, \dots, M, \tag{55}$$

*where $\Xi_j$ is the symmetric matrix composed of*

$$\begin{aligned}
\Xi_{11}^j &= P_jD_j + D_j^TP_j + 2\alpha_jP_j + H_1^T(r_j^2Q_{0j} + h_j^2W_{0j})H_1 \\
&\quad + \sum_{i=1}^{n-2} i^2 H_{i+1}^T\left(h_j^2\left(R_{ij} + e^{2\alpha_j ih_j}W_{ij}\right) + r_j^2e^{2\alpha_j ih_j}Q_{ij}\right)H_{i+1}, \\
\Xi_{12}^j &= \Xi_{18}^j = P_j\bar{B}_j[\bar{K}_{1j}, \dots, \bar{K}_{(n-1)j}], \quad \Xi_{15}^j = \mathrm{row}_{l=1,\dots,M,l\neq j}\{P_jF_{lj}\}, \\
\Xi_{16}^j &= P_j\bar{B}_j[\bar{K}_{0j}, \dots, \bar{K}_{(n-1)j}], \quad \Xi_{17}^j = P_j\bar{B}_j\bar{K}_{0j}, \quad \Xi_{19}^j = D_j^TH_{n-1}^T\bar{\Lambda}_j, \\
\Xi_{22}^j &= -\mathrm{diag}\{e^{-2\alpha_j ih_j}R_{ij}\}_{i=1}^{n-1}, \quad \Xi_{33}^j = -e^{-2\alpha_j r_j}Q_{0j} - \frac{\pi^2}{4}e^{-2\alpha(h_j+r_j)}W_{0j} \\
\Xi_{29}^j &= \Xi_{89}^j = [\bar{K}_{1j}, \dots, \bar{K}_{(n-1)j}]^T\bar{B}_j^TH_{n-1}^T\bar{\Lambda}_j, \quad \Xi_{37}^j = \frac{\pi^2}{4}e^{-2\alpha_j(h_j+r_j)}W_{0j}, \\
\Xi_{44}^j &= -\mathrm{diag}\{e^{-2\alpha_j r_j}Q_{ij} - \frac{\pi^2}{4}e^{-2\alpha_j(h_j+r_j)}W_{ij}\}_{i=1}^{n-1}, \\
\Xi_{48}^j &= \frac{\pi^2}{4}e^{-2\alpha_j(h_j+r_j)}[W_{ij}, \dots, W_{(n-1)j}]^T, \quad \Xi_{55}^j = -\mathrm{diag}_{l=1,\dots,M,l\neq j}\{2\beta_{lj}P_l\}, \\
\Xi_{59}^j &= \mathrm{row}_{l=1,\dots,M,l\neq j}\{F_{lj}^TH_{n-1}^T\bar{\Lambda}_j\}, \quad \Xi_{69}^j = [\bar{K}_{0j}, \dots, \bar{K}_{(n-1)j}]^T\bar{B}_j^TH_{n-1}^T\bar{\Lambda}_j, \\
\Xi_{66}^j &= -\mathrm{diag}\{\gamma_{ij}^2I\}_{i=0}^{n-1}, \quad \Xi_{77}^j = -\frac{\pi^2}{4}e^{-2\alpha_j(h_j+r_j)}W_{0j}, \quad \Xi_{79}^j = \bar{K}_{0j}^T\bar{B}_j^TH_{n-1}^T\bar{\Lambda}_j, \\
\Xi_{88}^j &= -\frac{\pi^2}{4}e^{-2\alpha_j(h_j+r_j)}\mathrm{diag}\{W_{ij}\}_{i=1}^{n-1}, \quad \Xi_{99}^j = -\bar{\Lambda}_j
\end{aligned} \tag{56}$$

*and other blocks are zero matrices. Here $D_j$ is given by (4) and*

$$\bar{\Lambda}_j = (n-1)^2\left(h_j^2\left(R_{(n-1)j} + e^{2\alpha_j(n-1)h_j}W_{(n-1)j}\right) + r_j^2e^{2\alpha_j(n-1)h_j}Q_{(n-1)j}\right). \tag{57}$$

*Then solution of subsystem (3) under the sampled-data delayed feedback (50) with controller gains (9) satisfies (32), where the $j$th component of the initial condition $z((n-1)\bar{h} + \bar{r}) \in \mathbb{R}^M$ is larger than $\bar{V}_j((n-1)\bar{h} + \bar{r})$ defined in (66) below for all $j = 1, \dots, M$. Moreover, from (16), given $\Delta = \sum_{j=1}^M \left(\sum_{i=0}^{n-1}\gamma_{ij}^2\bar{\omega}_{ij}^2\right)^2$ the ellipsoid (34) is exponentially attractive with a delay rate $\delta$ for all $x_j(t_0) \in \mathbb{R}^{nk_j}$ and $(n-1)$-times continuously differentiable and uniformly bounded $\omega_j(t)$ together with their derivatives, i.e. $|\omega_j^{(i)}(t)| \leq \bar{\omega}_{ij}$ $(i = 0, \dots, n-1)$ for all $t \geq 0$.*

*(ii) Given any $\alpha_j \in (0, \bar{\alpha}_j)$ and $\beta_{lj} > 0$ $(l = 1, \dots, M,\ l \neq j)$ such that the Metzler matrix $\mathcal{M}$ defined by (27) is Hurwitz, LMIs of item (i) are always feasible for small enough $r_j > 0$, $h_j > 0$, $\|F_{lj}\|$ $(l \neq j)$, $\gamma_{ij}^{-1} > 0$ $(i = 1, \dots, n-1)$, $\Gamma^{-1} > 0$ and $\delta > 0$.*

**Proof.** (i) Choose $V_{0j}(t)$ given by (26). Differentiating $V_{0j}(t)$ along the closed-loop subsystem (53) we have

$$\begin{aligned}
\dot{V}_{0j}(t) &= 2x_j^T(t)P_j[D_jx_j(t) + \bar{B}_j\bar{K}_{0j}(\delta_{0j}(t) + v_{0j}(s_k^j)) \\
&\quad + \sum_{i=1}^{n-1}\bar{B}_j\bar{K}_{ij}(\rho_{ij}(t) + \delta_{ij}(t) + v_{ij}(s_k^j)) + \sum_{l=1,l\neq j}^{M}F_{lj}x_l(t)].
\end{aligned} \tag{58}$$

We use the term $V_{\rho_{ij}}(t)$ given by (26) to compensate $\rho_{ij}(t)$ in (58). To compensate $\delta_{0j}$ in (58), we employ

$$\begin{aligned}
V_{\delta_{0j}}(t) &= h_j^2\int_{s_k^j}^t e^{-2\alpha_j(t-s)}x_{1j}^T(s)W_{0j}x_{1j}(s)ds \\
&\quad - \frac{\pi^2}{4}e^{-2\alpha_jh_j}\int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)}\delta_{0j}^T(s)W_{0j}\delta_{0j}(s)ds, \\
&\quad 0 < W_{0j} \in \mathbb{R}^{k_j\times k_j}, \quad t \in [s_k^j + r_j, s_{k+1}^j + r_j).
\end{aligned} \tag{59}$$

Note that the term $V_{\delta_{0j}}(t)$ in (59) can be represented as a sum of the continuous in time term $h_j^2\int_{t-r_j}^t e^{-2\alpha_j(t-s)}x_{1j}^T(s)W_{0j}x_{1j}(s)ds \geq 0$ with the discontinuous one

$$\begin{aligned}
\bar{V}_{\delta_{0j}}(t) &= h_j^2\int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)}x_{1j}^T(s)W_{0j}x_{1j}(s)ds \\
&\quad - \frac{\pi^2}{4}e^{-2\alpha_jh_j}\int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)}\delta_{0j}^T(s)W_{0j}\delta_{0j}(s)ds
\end{aligned}$$

that will vanishes at $t = s_k^j + r_j$, $k \in \mathbb{N}_0$. Since $\dot{\delta}_{0j}(t) = -x_{1j}(t)$ and $\delta_{0j}(s_k^j) = 0$, Lemma 2 implies $\bar{V}_{\delta_{0j}}(t) \geq 0$. We obtain

$$\begin{aligned}
\dot{V}_{\delta_{0j}}(t) + 2\alpha_jV_{\delta_{0j}}(t) &= h_j^2x_{1j}^T(t)W_{0j}x_{1j}(t) \\
&\quad - \frac{\pi^2}{4}e^{-2\alpha_j(h_j+r_j)}\delta_{0j}^T(t-r_j)W_{0j}\delta_{0j}(t-r_j)
\end{aligned}$$

$$= h_j^2 x_{1j}^T(t) W_{0j} x_{1j}(t) - \frac{\pi^2}{4} e^{-2\alpha_j(h_j+r_j)} (\delta_{0j}(t) - \kappa_{0j}(t))^T W_{0j} (\delta_{0j}(t) - \kappa_{0j}(t)), \quad (60)$$

where we used the relation

$$\delta_{0j}(t-r_j) = -\int_{s_k^j}^{t-r_j} x_{1j}(s)ds = \underbrace{-\int_{s_k^j}^{t} x_{1j}(s)ds}_{=\delta_{0j}(t)} - \underbrace{\left(-\int_{t-r_j}^{t} x_{1j}(s)ds\right)}_{=\kappa_{0j}(t)}.$$

For the term $\delta_{ij}(t)$ in (58), we consider

$$V_{\delta_{ij}}(t) = \int_{s_k^j}^{t} e^{-2\alpha_j(t-s)} \int_{s-ih_j}^{s} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{s-ih_j}^{s} \dot{x}_{ij}(\theta)d\theta ds - \frac{\pi^2}{4} e^{-2\alpha_j h_j} \int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)} \delta_{ij}^T(s) W_{ij} \delta_{ij}(s) ds,$$

$$0 < W_{ij} \in \mathbb{R}^{k_j \times k_j}, \quad i = 1, \dots, n-1, \quad t \in [s_k^j + r_j, s_{k+1}^j + r_j). \quad (61)$$

Similarly, the term $V_{\delta_{ij}}(t)$ in (61) can be represented as a sum of the continuous in time term $\int_{t-r_j}^{t} e^{-2\alpha_j(t-s)} \int_{s-ih_j}^{s} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{s-ih_j}^{s} \dot{x}_{ij}(\theta)d\theta ds \geq 0$ with the discontinuous one

$$\bar{V}_{\delta_{ij}}(t) = \int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)} \int_{s-ih_j}^{s} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{s-ih_j}^{s} \dot{x}_{ij}(\theta)d\theta ds - \frac{\pi^2}{4} e^{-2\alpha_j h_j} \int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)} \delta_{ij}^T(s) W_{ij} \delta_{ij}(s) ds$$

$$\geq h_j^2 \int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)} \int_{s-ih_j}^{s} \psi_{ij}(s-\theta)\dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{s-ih_j}^{s} \psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds - \frac{\pi^2}{4} e^{-2\alpha_j h_j} \int_{s_k^j}^{t-r_j} e^{-2\alpha_j(t-s)} \delta_{ij}^T(s) W_{ij} \delta_{ij}(s) ds$$

that will vanishes at $t = s_k^j + r_j$, $k \in \mathbb{N}_0$. Since $\dot{\delta}_{ij}(t) = -\int_{t-ih_j}^{t} \psi_{ij}(t-\theta)\dot{x}_{ij}(\theta)d\theta$ and $\delta_{ij}(s_k^j) = 0$, Lemma 2 implies $\bar{V}_{\delta_{1j}}(t) \geq 0$. We obtain

$$\dot{V}_{\delta_{ij}}(t) + 2\alpha_j V_{\delta_{ij}}(t) = \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{t-ih_j}^{t} \dot{x}_{ij}(\theta)d\theta - \frac{\pi^2}{4} e^{-2\alpha_j(h_j+r_j)} \delta_{ij}^T(t-r_j) W_{ij} \delta_{1j}(t-r_j). \quad (62)$$

To compensate the term $\int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{t-ih_j}^{t} \dot{x}_{ij}(\theta)d\theta$ in the above expression, we additionally consider (Selivanov & Fridman, 2018b)

$$\tilde{V}_{\delta_{ij}}(t) = ih_j e^{2\alpha_j ih_j} \int_{t-ih_j}^{t} e^{-2\alpha_j(t-\theta)} (\theta - t + ih_j) \dot{x}_{ij}^T(\theta) W_{ij} \dot{x}_{ij}(\theta) d\theta,$$

$$i = 1, \dots, n-1. \quad (63)$$

Thus,

$$\dot{\tilde{V}}_{\delta_{ij}}(t) + 2\alpha_j \tilde{V}_{\delta_{ij}}(t) = (ih_j)^2 e^{2\alpha_j ih_j} \dot{x}_{ij}^T(t) W_{ij} \dot{x}_{ij}(t) - ih_j e^{2\alpha_j ih_j} \int_{t-ih_j}^{t} e^{-2\alpha_j(t-\theta)} \dot{x}_{ij}^T(\theta) W_{ij} \dot{x}_{ij}(\theta) d\theta$$

$$\leq (ih_j)^2 e^{2\alpha_j ih_j} \dot{x}_{ij}^T(t) W_{ij} \dot{x}_{ij}(t) - \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{t-ih_j}^{t} \dot{x}_{ij}(\theta)d\theta, \quad (64)$$

where we used Jensen's inequality (see (3.87) in Fridman (2014))

$$ih_j \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta) W_{ij} \dot{x}_{ij}(\theta) d\theta \geq \int_{t-ih_j}^{t} \dot{x}_{ij}^T(\theta)d\theta W_{ij} \int_{t-ih_j}^{t} \dot{x}_{ij}(\theta)d\theta.$$

Taking into account the following

$$\delta_{ij}(t-r_j) = -\int_{s_k^j}^{t-r_j} \int_{s-ih_j}^{s} \psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds = \underbrace{-\int_{s_k^j}^{t} \int_{s-ih_j}^{s} \psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds}_{=\delta_{ij}(t)} - \underbrace{\left(-\int_{t-r_j}^{t} \int_{s-ih_j}^{s} \psi_{ij}(s-\theta)\dot{x}_{ij}(\theta)d\theta ds\right)}_{=\kappa_{ij}(t)}$$

and using (62), (64), we have

$$\dot{V}_{\delta_{ij}}(t) + 2\alpha_j V_{\delta_{ij}}(t) + \dot{\tilde{V}}_{\delta_{ij}}(t) + 2\alpha_j \tilde{V}_{\delta_{ij}}(t) \leq (ih_j)^2 e^{2\alpha_j ih_j} \dot{x}_{ij}^T(t) W_{ij} \dot{x}_{ij}(t) - \frac{\pi^2}{4} e^{-2\alpha_j(h_j+r_j)} (\delta_{ij}(t) - \kappa_{ij}(t))^T W_{ij} (\delta_{ij}(t) - \kappa_{ij}(t)). \quad (65)$$

Moreover, we use the terms $V_{\kappa_{ij}}(t)$ ($i = 0, \dots, n-1$) given by (26) to compensate $\kappa_{ij}(t)$.

We now consider the following Lyapunov functional:

$$\tilde{V}_j(t) = V_j(t) + V_{\delta_{0j}}(t) + \sum_{i=1}^{n-1} \left( V_{\delta_{ij}}(t) + \tilde{V}_{\delta_{ij}}(t) \right), \quad (66)$$

where $V_j(t)$, $V_{\delta_{0j}}(t)$, $V_{\delta_{ij}}(t)$ and $\tilde{V}_{\delta_{ij}}(t)$ are, respectively, from (25), (59), (61) and (63). From (36)–(42), (58), (60) and (65), we find

$$\dot{\tilde{V}}_j(t) + 2\alpha_j \tilde{V}_j(t) - 2 \sum_{l=1, l\neq j}^{M} \beta_{lj} x_l^T(t) P_l x_l(t) - \sum_{i=0}^{n-1} \gamma_{ij}^2 |v_{ij}(s_k^j)|^2 \leq \tilde{\zeta}_j^T(t) \tilde{\Xi}_j \tilde{\zeta}_j(t) + \dot{x}_{(n-1)j}^T \bar{\Lambda}_j \dot{x}_{(n-1)j}, \quad (67)$$

where $\tilde{\Xi}_j$ is obtained from $\Xi_j$ (composed of (56)) by taking away the last block-column and block-row, $\bar{\Lambda}_j$ is given by (57), and

$$\tilde{\zeta}_j(t) = \text{col}\{\bar{\zeta}_j(t), v_{0j}(s_k^j), \dots, v_{(n-1)j}(s_k^j), \delta_{0j}(t), \dots, \delta_{(n-1)j}(t)\}$$

with $\bar{\zeta}_j(t)$ defined in (44). Substituting $\dot{x}_{(n-1)j}(t) = H_{n-1}\dot{x}_j(t)$ with $\dot{x}_j(t)$ satisfying (53) into (67) and then applying Schur's complement lead to (45) with $V_j(t)$ and $V_l$ changed by $\tilde{V}_j(t)$ and $\tilde{V}_l(t)$, respectively. Then following arguments of Theorem 1, one can find the ISS of the subsystem with an ellipsoid given by (34) is ensured provided LMIs in (55) holds.

(ii) The proof is similar to the proof of Theorem 1(ii). □

## 4. Examples

**Example 1** ($M = 1$)**.** Consider (1) with

$$A_{i1} = 0, \quad i = 0, 1, 2, \quad B_1 = 1. \quad (68)$$

Under the continuous state-feedback (5), as in Selivanov and Fridman (2018b) we choose the controller gains

$$\bar{K}_{01} = -2 \times 10^{-4}, \quad \bar{K}_{11} = -0.06, \quad \bar{K}_{21} = -0.342. \quad (69)$$

By solving LMIs of Theorem 1 with $\alpha_1 = 0$ and different values of input delays $r_1$, we find the solutions that guarantee the stability of system (1), (5), (68), (69): $r_1 = 0$, $h_1 = 2.529$; $r_1 = 0.01$, $h_1 = 2.509$. Clearly, in the case of $r_1 = 0$ our conditions lead to the same result as Selivanov and Fridman (2018b). Note that the case of $r_1 \neq 0$ was not considered in Selivanov and Fridman (2018b).

**Example 2** ($M = 2$)**.** Consider two coupled inverted pendulums on two carts (Freirich & Fridman, 2016; Heemels et al., 2013) described by (1) with

$$A_{0j} = \begin{bmatrix} 2.9156 & -0.0005 \\ -1.6663 & 0.0002 \end{bmatrix}, \quad A_{1j} = F_{lj1} = 0, \quad F_{lj0} = \vartheta_j \tilde{F}$$

$$B_j = \begin{bmatrix} -0.0042 \\ 0.0167 \end{bmatrix}, \quad j = 1, 2, \quad \tilde{F} = \begin{bmatrix} 0.11 & 0.05 \\ -0.03 & -0.02 \end{bmatrix} \quad (70)$$

and choose as in Freirich and Fridman (2016), Heemels et al. (2013) the controller gains

$$[\bar{K}_{01} \quad \bar{K}_{11}] = \begin{bmatrix} 11396 & 573.96 & 7196.2 & 1199.0 \end{bmatrix},$$

$$[\bar{K}_{02} \quad \bar{K}_{12}] = \begin{bmatrix} 29241 & 2875.3 & 18135 & 3693.9 \end{bmatrix}. \quad (71)$$

It is clear that with the above gains, matrix $D_j$ defined by (4) is Hurwitz. Therefore, the derivative-dependent feedback (5) with these gains from (71) stabilizes system (1), (70) for small enough $r_j > 0$ and $\|F_{lj}\|$, where $F_{lj}$ is defined in (2).

Let $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\gamma_{12} = 0.1$, $r_1 = r_2 = 0$, $h_1 = 0.02$ and $h_2 = 0.12$. We now make a comparison between the vector Lyapunov method and the existing method (Freirich & Fridman, 2016; Zhu & Fridman, 2020b):

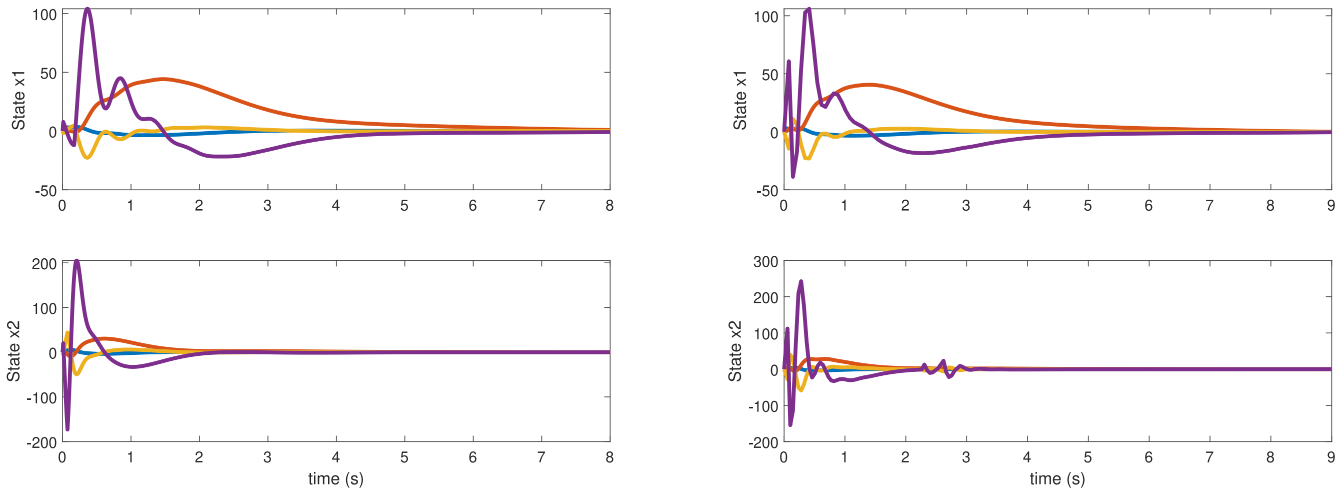


**Fig. 1.** State trajectories of system (1), (70), where $r_1 = r_2 = 0.01$, under the delay-dependent feedback (12), (71) (left) and the sampled-data controller (50), (71) (right).

(i) First, we find results via the method of Freirich and Fridman (2016), Zhu and Fridman (2020b) as explained in Remark 3. From Remark 3 it follows that $\gamma_{21}$ should be less than 0.5, where in this example we choose $\gamma_{21} = 0.49$ leading to the system decay rate $2 \times \min\{0.5 - 0.49, 0.5 - 0.1\} = 0.02$. We then verify (29) to find the maximum values $\vartheta_1 = 6.01$, $\vartheta_2 = 1.01$ preserving the exponential stability (thus, ISS) of system (1), (5), (70), (71) with decay rate 0.02.

(ii) Second, we find results via the vector Lyapunov method. From (27) it follows that $\beta_{21}$ should be less than 2.5, where in this example we choose $\beta_{21} = 2.35$. We then verify (28), (29) with $\delta = 0.03$ to find the maximum values $\vartheta_1 = 8.78$, $\vartheta_2 = 1.5$ preserving the exponential stability (thus, ISS) of system (1), (5), (70), (71) with decay rate 0.03.

Clearly, the vector Lyapunov method allows larger coupling and larger decay rate than those via the method of Freirich and Fridman (2016), Zhu and Fridman (2020b) when $\beta_{12}$ is small. Note also that if $\beta_{12}$ is not small, e.g. $\beta_{12} = 0.49$ (that is slightly smaller than $\alpha_1 = 0.5$), both methods lead to the same $\beta_{21} = 0.49$ and thus, the same coupling.

We now consider system (1), (70) under the delay-dependent feedback (12), (71). By verifying (29) with $\delta = 0.01$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\beta_{12} = 0.1$, $\beta_{21} = 2.35$, $\vartheta_1 = 5.5$, $\vartheta_2 = 0.5$, we find the solutions (see lines 2–4 of Table 1) that guarantee the exponential stability (thus, ISS) of the system (1), (5), (70), (71).

We next consider system (1), (70) under the sampled-data controller (71), (50), where $M = 2$. By solving LMIs of Theorem 2 with $\delta = 0.01$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\beta_{12} = 0.1$, $\beta_{21} = 2.35$, $\vartheta_1 = 5.5$, $\vartheta_2 = 0.5$, we find the solutions (see lines 2–4 of Table 2) that guarantee the exponential stability (thus, ISS) of the system (1), (50) (70), (71).

Finally, choose the initial condition $x_j(0) = [\pi, 0, 0, 0]^T$ and the disturbance $\omega_j(t) = [0.01\sin(t), 0]^T$. Simulation results presented in Fig. 1 show that the stability of system (1), (70) under the delay-dependent feedback (12), (71) with $h_1 = 0.16$, $h_2 = 0.062$ and under the sampled-data delayed controller (50), (71) with $h_1 = 0.067$, $h_2 = 0.045$ are guaranteed, where $r_1 = r_2 = 0.01$.

**Example 3** ($M = 3$). Consider (1) with $A_{0j}$, $A_{1j}$, $B_j$ and $F_{lj1}$ given by (70) and

$$F_{210} = \vartheta_1 \tilde{F}, \quad F_{120} = F_{320} = \vartheta_2 \tilde{F}, \quad F_{230} = \vartheta_3 \tilde{F}, \quad F_{310} = F_{130} = 0,$$

where $\tilde{F}$ is from (70). Choose the controller gains as (71) and

$$[\bar{K}_{03} \quad \bar{K}_{13}] = \begin{bmatrix} 20317.5 & 1724.6 & 12666 & 2446.4 \end{bmatrix}.$$

By solving LMIs of Theorems 1 and 2 with $\delta = 0.01$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\alpha_3 = 1$, $\beta_{12} = 0.1$, $\beta_{21} = 2.35$, $\beta_{32} = 0.01$, $\beta_{23} = 0.2$, $\vartheta_1 = 5.5$, $\vartheta_2 = 0.5$,

**Table 1**
Solutions under continuous delayed control.

| | $r_1$ | $r_2$ | $r_3$ | $h_1$ | $h_2$ | $h_3$ |
|---|---|---|---|---|---|---|
| $M = 2$ | 0 | 0 | – | 0.178 | 0.127 | – |
| | 0.005 | 0.015 | – | 0.169 | 0.098 | – |
| | 0.01 | 0.01 | – | 0.16 | 0.062 | – |
| $M = 3$ | 0 | 0 | 0 | 0.168 | 0.101 | 0.032 |
| | 0.01 | 0.01 | 0.01 | 0.153 | 0.021 | 0.024 |
| | 0.02 | 0.03 | 0.01 | 0.130 | 0.012 | 0.010 |

**Table 2**
Solutions under sampled-data delayed control.

| | $r_1$ | $r_2$ | $r_3$ | $h_1$ | $h_2$ | $h_3$ |
|---|---|---|---|---|---|---|
| $M = 2$ | 0 | 0 | – | 0.076 | 0.024 | – |
| | 0.05 | 0.015 | – | 0.071 | 0.041 | – |
| | 0.01 | 0.01 | – | 0.067 | 0.045 | – |
| $M = 3$ | 0 | 0 | 0 | 0.075 | 0.025 | 0.012 |
| | 0.005 | 0.005 | 0.005 | 0.062 | 0.010 | 0.008 |
| | 0.01 | 0.015 | 0.01 | 0.055 | 0.004 | 0.002 |

$\vartheta_3 = 1.2$, we find the efficient solutions (see lines 5–7 of Table 1 and of Table 2, respectively) that guarantee the exponential stability (thus, ISS) of the system.

## 5. Conclusions

We have given a constructive solution to decentralized derivative-dependent control of large-scale $n$th-order systems with input delays and unknown disturbances via delayed feedback implementation. This was done by extending the recent results in the centralized case and by adding appropriate terms to the corresponding Lyapunov functional for the compensation of the additional terms due to input delays. Note that the couplings under consideration are constant. Future research may focus on the time-varying couplings in large-scale systems.

## CRediT authorship contribution statement

**Jin Zhang:** Funding acquisition, Investigation, Writing – original draft, Writing – review & editing. **Hui Zhang:** Writing – original draft, Writing – review & editing. **Emilia Fridman:** Supervision.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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