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Sampled-data finite-dimensional boundary control of 2D semilinear parabolic stochastic PDEs

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***Abstract*—This paper addresses the sampled-data boundary stabilization of 2D semilinear parabolic stochastic PDEs with globally Lipschitz nonlinearities. We consider Dirichlet actuation and design a finite-dimensional state-feedback controller with the shape functions in the form of eigenfunctions corresponding to the first N comparatively unstable eigenvalues. We extend the trigonometric change of variables to the 2D case and further improve it that leads to a dynamic extension with the corresponding proportional-integral controller, where sampled-data control is implemented via a generalized hold device. By employing the corresponding Itô formulas for stochastic ODEs and PDEs, respectively, and suggesting a non-trivial stochastic extension of the descriptor method, we derive linear matrix inequalities (LMIs) for finding the controller dimension and gain that guarantees the global mean-square L^2 exponential stability for the full-order closed-loop system. A numerical example demonstrates the efficiency and advantage of our method.**

***Index Terms*—2D PDEs, semilinear stochastic heat equation, sampled-data control, boundary control.**

I. Introduction

Boundary control of PDEs is attractive from the theoretical and practical point of view. It can be designed by the backstepping approach [1] or modal decomposition [2]–[7]. Boundary control of 1D semilinear heat equation was suggested in [8] by using a modal decomposition method via dynamic extension. These results were improved in [9] via direct Lyapunov approach. In the present paper we will extend the results in [8], [9] to 2D semilinear heat equations and further improve them.

Sampled-data boundary finite-dimensional controllers for 1D parabolic PDEs, implemented by zero-order hold devices, were suggested in [10], [11] for state-feedback control, and in [12], [13] for observer-based control. For the finite-dimensional boundary control under point measurement of 1D linear heat equations, sampled-data control via a generalized hold device was suggested in [6]. Event-triggered boundary control of 1D heat equations was studied in [14], [15] via backstepping design. However, all these results were confined to 1D parabolic deterministic PDEs.

Control of stochastic PDEs has gained significant attention due to its broad applications in engineering and finance. In [16], finite-dimensional boundary control of 1D linear stochastic PDEs was studied, where constructive conditions for finding the controller dimension were not provided. In [17], we suggested a constructive finite-dimensional state- and output-feedback boundary controllers for 1D linear parabolic stochastic PDEs, extending results of [5] to the stochastic case. Inspired by [6], [13], we presented finite-dimensional sampled-data observer-based boundary controller [18] and sampled-data sub-predictors [19] for 1D linear and semilinear stochastic heat equations. However, these results remain limited to 1D case.

Boundary state-feedback stabilization of ND linear parabolic PDEs was studied in [20] via backstepping method and in [21], [22] via modal decomposition approach. Note that only specific classes of semilinear PDEs have been addressed through the backstepping method and the feedback design in [21], [22] is not applicable for semilinear case due to the spillover behavior caused by the nonlinearity (see [9]). In [23], [24], the finite-dimensional boundary regional state-feedback stabilization for ND semilinear parabolic PDEs was studied via a fixed point argument. Following [24], finite-dimensional observer-based boundary control was studied in [25] for 2D and 3D linear parabolic PDEs. In our recent paper [26], sampled-data and delayed finite-dimensional observer-based design for 2D linear deterministic heat equation was studied under Neumann actuation, where as in 1D case (see [27]) dynamic extension was not needed. Dirichlet boundary control and its sampled-data implementation for high-dimensional semilinear parabolic deterministic and stochastic PDEs remains an open problem. The main challenges lie in the following: (i) finding an efficient dynamic extension in the presence of multiple eigenvalues; (ii) managing with complicated stability analysis due to slower convergence of the eigenvalues to infinity; (iii) providing an efficient controller design in the presence of multiplicative noise and delays.

In this paper, we study the sampled-data finite-dimensional state-feedback global stabilization of 2D semilinear parabolic stochastic PDEs under Dirichlet actuation, where the nonlinearities satisfy globally Lipschitz condition and network-induced transmission delays are considered. To address the controllability issues caused by the multiple eigenvalues, we design the boundary controller with the shape functions in the form of eigenfunctions corresponding to the first N eigenvalues. Such shape functions were previously considered in [21], [23], [24]. We construct a Lyapunov functional that depends on both the deterministic and stochastic parts of the finite-dimensional part of the closed-loop system. By employing the corresponding Itô formulas for stochastic ODEs and PDEs, respectively, and using the descriptor method (a stochastic extension of [28, Sec. 5]), we provide LMI conditions for finding the controller dimension and gain, as well as upper

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bounds on sampling intervals and Lipschitz constants that preserve the mean-square L^2 exponential stability. The main contribution of this paper is listed as follows:

- For the first time, we present constructive results for the control of 2D semilinear heat equation under the Dirichlet actuation with efficient quantitative bounds on the number of modes used in the controller, whereas controller gain is found from the design LMIs.
- We efficiently extend the change of variables and dynamic extension from the 1D case in [8], [9] to the 2D case and further improve it by suggesting an additional tuning parameter that always enlarges the Lipschitz bounds for 2D and 1D PDEs both in deterministic and stochastic cases - achieving an increase of over 50% in some cases in our example.
- Differently from [18] and [26] where controller gain is designed from the corresponding deterministic case and non-delayed case, respectively, we propose a stochastic delayed extension of the descriptor method that essentially enlarges the sampling intervals and transmission delays - exceeding an increase of over 100% in our example. Moreover, sampled-data boundary control under Dirichlet actuation is for the first time considered in the presence of additional variable delay.

Preliminary results on continuous-time stabilization were reported in ECC 2024 [29].

Notations: Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub σ-fields of $\mathcal{F}$ and let $\mathbb{E}\{\cdot\}$ be the expectation operator. Denote by $\mathcal{W}(t)$ the 1D standard Brownian motion defined on $(\Omega,\mathcal{F},\mathbb{P})$. The Euclidean norm is denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is positive definite with symmetric elements denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^{\mathrm{T}}Px$. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{Z}_+$ be the set of nonnegative integers. For any bounded domain $\mathcal{O}_0\subset\mathbb{R}^n$ ($n=1,2$), denote by $L^2(\mathcal{O}_0)$ the space of square integrable functions with inner product $\langle f,g\rangle_{\mathcal{O}_0}=\int_{\mathcal{O}_0}f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|^2_{L^2(\mathcal{O}_0)}=\langle f,f\rangle_{\mathcal{O}_0}$. $H^1(\mathcal{O}_0)$ is the Sobolev space of functions $f:\mathcal{O}_0\longrightarrow\mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(\mathcal{O}_0)$ is $\|f\|^2_{H^1(\mathcal{O}_0)}=\|f\|^2_{L^2(\mathcal{O}_0)}+\|\nabla f\|^2_{L^2(\mathcal{O}_0)}$, where ∇f represents the gradient of f and $\|\nabla f\|^2_{L^2(\mathcal{O}_0)}=\int_{\mathcal{O}_0}|\nabla f(x)|^2\mathrm{d}x$. Let $\frac{\partial}{\partial\mathbf{n}}$ be the normal derivative.

II. Problem description

A. System under consideration

Let $\mathcal{O}\subset\mathbb{R}^2$ be a bounded open connected set. We assume that either the boundary $\partial\mathcal{O}=\Gamma_1\cup\Gamma_2$ ($\Gamma_1\cap\Gamma_2$) is of class C^2 or $\mathcal{O}$ is a rectangular domain. Consider the following 2D semilinear stochastic heat equation:

$$\begin{aligned}&\mathrm{d}z(x,t)=[\Delta z(x,t)+qz(x,t)+f(z(x,t))]\mathrm{d}t\\&\qquad+g(z(x,t))\mathrm{d}\mathcal{W}(t),\quad x\in\mathcal{O},\\&z(x,t)=0,\ x\in\Gamma_1,\quad z(x,t)=u(x,t),x\in\Gamma_2,\end{aligned}\tag{1}$$

where Δ is the usual Laplacian, $q\in\mathbb{R}$ is the reaction coefficient, $u(x,t)$ is the control input to be designed, $f,g:\mathbb{R}\to\mathbb{R}$ are globally Lipschitz functions satisfying

$$\begin{aligned}&f(0)=0,|f(z_1)-f(z_2)|\leq\sigma_f|z_1-z_2|,\\&g(0)=0,|g(z_1)-g(z_2)|\leq\sigma_g|z_1-z_2|,\forall z_1,z_2\in\mathbb{R},\end{aligned}\tag{2}$$

for some $\sigma_f,\sigma_g>0$. Here σ_g defines the upper bound of noise intensity.

Let

$$\begin{aligned}&\mathcal{A}\phi=-\Delta\phi,\ \mathcal{D}(\mathcal{A})=\{\phi|\phi\in H^2(\mathcal{O})\cap H_0^1\},\\&H_0^1=\{\phi\in H^1(\mathcal{O})|\phi(x)=0\text{ for }x\in\partial\mathcal{O}\}.\end{aligned}\tag{3}$$

It follows from [30, Proposition 3.2.12] that the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of $\mathcal{A}$ are real and we can repeat each eigenvalue according to its finite multiplicity to get

$$\lambda_1<\lambda_2\leq\cdots\leq\lambda_n\leq\dots,\quad\lim_{n\to\infty}\lambda_n=\infty.\tag{4}$$

We denote the corresponding eigenfunctions as $\{\phi_n\}_{n=1}^\infty$. For λ_n, we have the following estimate which will be used for the asymptotic feasibility of LMIs:

***Lemma** 1:* (see [31, Sec. 11.6]) For eigenvalues (4), we have $\lim_{n\to\infty}\frac{\lambda_n}{n}=\frac{4\pi}{|\mathcal{O}|}$, where $|\mathcal{O}|$ is the area of $\mathcal{O}$.

***Remark** 1:* We assume that the eigenvalues and eigenfunctions are explicitly known. However, in many practical applications, only approximate knowledge of the eigenfunction structure is available. Addressing unknown eigenvalues and eigenfunctions through approximate estimation will be a potential direction for future research.

Let $\delta>0$ be a desired decay rate and $N\in\mathbb{N}$ such that

$$-\lambda_n+q+\delta+\sqrt{2}\sigma_f+\sigma_g^2<0,\quad n>N,\tag{5}$$

where N denotes the number of unstable modes and will be determined by $\chi_n<0$, $n>N$ in (45a) below. Our controller will be designed by using N modes. For given $\lambda\in\{\lambda_n\}_{n=1}^N$, let m_λ be the geometric multiplicity of λ and $\phi_\lambda^{(1)},\dots,\phi_\lambda^{(m_\lambda)}$ be the eigenfunctions corresponding to λ. We impose the following assumption that is crucial for the controllability of the finite-dimensional part of the closed-loop system (see above (22)):

***Assumption** 1:* Given $\lambda\in\{\lambda_n\}_{n=1}^N$, let $\{\frac{\partial\phi_\lambda^{(i)}}{\partial\mathbf{n}}\}_{i=1}^{m_\lambda}$ be linearly independent in $L^2(\Gamma_2)$.

***Remark** 2:* Assumption 1 always holds true for 1D case (due to simple eigenvalues) and for rectangular domain $\mathcal{O}=(0,a_1)\times(0,a_2)$, $a_1,a_2>0$. Consider the boundary:

$$\partial\mathcal{O}=\Gamma_1\cup\Gamma_2,\quad\Gamma_2=\{(x_1,0),\ x_1\in(0,a_1)\}.\tag{6}$$

Here the eigenvalues of $\mathcal{A}$ are given by

$$\lambda_{m,k}=\pi^2[\tfrac{m^2}{a_1^2}+\tfrac{k^2}{a_2^2}],\quad m,k\in\mathbb{N},\tag{7}$$

whereas the corresponding eigenfunctions have the form

$$\phi_{m,k}(x)=\tfrac{2}{\sqrt{a_1a_2}}\sin(\tfrac{m\pi x_1}{a_1})\sin(\tfrac{k\pi x_2}{a_2}),x=(x_1,x_2).\tag{8}$$

For any pair of multiple eigenvalues $\lambda_{m_1,k_1}=\lambda_{m_2,k_2}$, the relation $m_1\neq m_2$ always implies $k_1\neq k_2$ (and vice versa). Therefore, $\frac{\partial\phi_{m_1,k_1}}{\partial\mathbf{n}}$ and $\frac{\partial\phi_{m_2,k_2}}{\partial\mathbf{n}}$ are always linearly independent in $L^2(\Gamma_2)$. Note that Assumption 1 is much weaker than the assumption (ii) in [23] (linear independence

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of $\{\frac{\partial \phi_n(x)}{\partial \mathbf{n}}, x \in L^2(\Gamma_2)\}_{n=1}^N$), which does not hold for square domain when $N \geq 3$ (see Remark 3.2 in [23]). Assumption (ii) in [23] was removed in [24], [25] by slightly perturbing the linear operator $\mathcal{A}$, whereas constructive conditions for finding the controller dimension were not provided.

Remark 3: In [26], the sampled-data and delayed observer-based design for 2D linear heat equation was explored under Neumann actuation, which is not applicable for Dirichlet actuation (similar to the 1D case explained in [19, Remark 2]). In this paper, we manage with the Dirichlet actuation via dynamic extension and the results can be directly extended to the Neumann actuation. In this scenario, we do not need Assumption 1 since for general domain $\mathcal{O}$ and given $\lambda \in \{\lambda_n\}_{n=1}^N$, eigenvectors $\{\phi_\lambda^{(i)}\}_{i=1}^{m_\lambda}$ are always linearly independent in $L^2(\Gamma_2)$ (see [21, Lemma 7.1]).

B. Change of variables

For given positive constants $\{\mu_i\}_{i=1}^N$ satisfying

$$\mu_i \neq \lambda_n, \quad i \in \{1, \ldots, N\}, n \in \mathbb{Z}, \quad \mu_i = O(\lambda_i), \tag{9}$$

consider functions $\psi_i \in L^2(\mathcal{O})$, $i = 1, \ldots, N$, that satisfy

$$\begin{aligned} &\Delta\psi_i(x) = -\mu_i\psi_i(x), \quad x \in \mathcal{O}, \\ &\psi_i(x) = 0, \ x \in \Gamma_1, \quad \psi_i(x) = b_i \tfrac{\partial \phi_i(x)}{\partial \mathbf{n}}, \ x \in \Gamma_2, \end{aligned} \tag{10}$$

where $b_i \in \mathbb{R}$ are chosen such that $\|\psi_i\|_{L^2(\mathcal{O})} = \rho$. Here $\rho > 0$ is a tuning parameter. Such functions always exist [21]. Since $\mu_i \neq \lambda_n$, by applying Green's first identity, we find that

$$\langle \psi_i, \phi_n \rangle_{\mathcal{O}} = \tfrac{-b_i}{\lambda_n - \mu_i} \langle \tfrac{\partial \phi_i}{\partial \mathbf{n}}, \tfrac{\partial \phi_n}{\partial \mathbf{n}} \rangle_{\Gamma_2}. \tag{11}$$

Remark 4: For rectangular domain introduced in Remark 2, we take

$$\begin{aligned} &\psi_{m,k}(x) = \tfrac{2\rho}{\sqrt{a_1 a_2}} \sin(\tfrac{m\pi x_1}{a_2}) \cos(\tfrac{(k-0.5)\pi x_2}{a_2}), \\ &\mu_{m,k} = (\tfrac{m\pi}{a_1})^2 + (\tfrac{(k-0.5)\pi}{a_2})^2, \ b_{m,k} = -\tfrac{\rho a_2}{k\pi}, \end{aligned} \tag{12}$$

where $m, k \in \mathbb{N}$. We reorder the eigenvalues (7) to form a non-decreasing sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying (4) and denote the corresponding eigenfunctions as $\{\phi_n\}_{n=1}^\infty$. Following the corresponding relationship between (7) and (4), we reorder $\{\psi_{m,k}\}_{m,k=0}^\infty$, $\{\mu_{m,k}\}_{m,k=0}^\infty$, and $\{b_{m,k}\}_{m,k=0}^\infty$ as $\{\psi_i\}_{i=1}^\infty$, $\{\mu_i\}_{i=1}^\infty$, and $\{b_i\}_{i=1}^\infty$. We see that $\{\mu_i\}_{i=1}^N$ satisfy (9), $\{\psi_i\}_{i=1}^\infty$ satisfy (10), and $\|\psi_i\|_{L^2(\mathcal{O})} = \rho$, $i \in \mathbb{Z}_+$.

Remark 5: Functions ψ_i in (10) are the extension of the 1D functions considered in [8], [9] where ρ is fixed to $\frac{1}{\sqrt{2}}$. In our example in Sec IV, we show that an appropriate choice of $\rho \in (0, \frac{1}{\sqrt{2}})$ can lead to larger upper bounds on the Lipschitz constants and sampling intervals. Moreover, the tuning parameter ρ allows also to improve results in the 1D case (see the conference version [29]).

We design the control input with the shape functions in the form of eigenfunctions $\{\phi_i\}_{i=1}^N$:

$$u(x,t) = \textstyle\sum_{i=1}^N b_i \frac{\partial \phi_i(x)}{\partial \mathbf{n}} u_i(t), \ x \in \Gamma_2, \tag{13}$$

where $u_i(t)$, $i = 1, \ldots, N$ are to be designed later. Consider the change of variables:

$$\begin{aligned} &w(x,t) = z(x,t) - \psi^{\mathrm{T}}(x)\mathbf{u}(t), \\ &\psi(x) = \mathrm{col}\{\psi_i(x)\}_{i=1}^N, \quad \mathbf{u}(t) = \mathrm{col}\{u_i(t)\}_{i=1}^N. \end{aligned} \tag{14}$$

Substituting (14) into (1) we obtain

$$\begin{aligned} \mathrm{d}w(x,t) = {} & [\Delta w(x,t) + qw(x,t) + \psi^{\mathrm{T}}(x)\Xi_0\mathbf{u}(t) \\ & + f(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))]\mathrm{d}t - \psi^{\mathrm{T}}(x)\mathrm{d}\mathbf{u}(t) \\ & + g(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))\mathrm{d}\mathcal{W}(t), \ x \in \mathcal{O}, \\ w(x,t) = {} & 0, \ x \in \partial\mathcal{O}, \quad w(x,0) = z(x,0), \end{aligned} \tag{15}$$

where $\Xi_0 = \mathrm{diag}\{-\mu_1 + q, \ldots, -\mu_N + q\}$.

C. Network-based controller design

As shown in Fig. 1, we consider network-based control in the presence of communication delay from controller to actuator. We denote by $\{s_j\}_{j=0}^\infty$ sampling instants on the controller side with $0 = s_0 < s_1 < \cdots < s_j < \ldots$, $\lim_{j\to\infty} s_j = \infty$, $s_{j+1} - s_j \leq h$, where $h > 0$ is the maximum sampling interval. Denote by t_j the updating time of the actuator, and suppose that the updating signal at the instant t_j has experienced a controller-to-actuator transmission delay η_j. Note that $\eta_j \geq 0$ is varying at different instants t_j and is upper bounded by a known constant $\eta_M > 0$. Then we have the updating time of actuator $\{t_j\}_{j=0}^\infty$ satisfying

$$t_j = s_j + \eta_j, \quad 0 < t_{j+1} - t_j \leq h + \eta_M =: \tau_M.$$

Using the time-delay approach to sampled-data control (see [28]), we introduce the following representation of delay:

$$\tau(t) = t - t_j + \eta_j, \ t \in [t_j, t_{j+1}), \quad \tau(t) \leq \tau_M. \tag{16}$$

Henceforth the dependence of $\tau(t)$ on t will be suppressed to shorten notations.

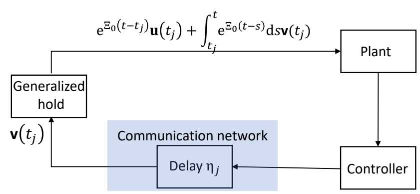

Fig. 1. Sampled-data control of a heat equation.

In (15), we treat $\mathbf{u}(t)$ as an additional state variable, which is generated by a generalized hold device:

$$\begin{aligned} &\mathrm{d}\mathbf{u}(t) = [\Xi_0\mathbf{u}(t) + \mathbf{v}(t_j)]\mathrm{d}t, \quad t \in [t_j, t_{j+1}), \\ &\mathbf{u}(t) = 0, t \in [0, t_0], \ u(t_j) = \lim_{t \to t_j^-} u(t), j \geq 1, \end{aligned} \tag{17}$$

where the values $\{\mathbf{v}(t_j)\}_{j=1}^\infty$ are control inputs to be determined. Given $\mathbf{v}(t_j)$, $\mathbf{u}(t)$ is calculated as

$$\mathbf{u}(t) = \mathrm{e}^{\Xi_0(t-t_j)}\mathbf{u}(t_j) + \textstyle\int_{t_j}^t \mathrm{e}^{\Xi_0(t-s)}\mathrm{d}s\mathbf{v}(t_j), t \in [t_j, t_{j+1}).$$

From (15) and (17), we have the following equivalent system:

$$\begin{aligned} &\mathrm{d}w(x,t) = [\Delta w(x,t) + qw(x,t) - \psi^{\mathrm{T}}(x)\mathbf{v}(t_j) \\ &\quad + f(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))]\mathrm{d}t \\ &\quad + g(w(x,t) + \psi^{\mathrm{T}}(x)\mathbf{u}(t))\mathrm{d}\mathcal{W}(t), \ t \in [t_j, t_{j+1}), \\ &w(x,t)|_{x\in\partial\mathcal{O}} = 0, w(x,0) = z(x,0). \end{aligned} \tag{18}$$

Present the solution to (18) as

$$w(x,t) = \textstyle\sum_{n=1}^\infty w_n(t)\phi_n(x), \ w_n(t) = \langle w(\cdot,t), \phi_n \rangle_{\mathcal{O}}. \tag{19}$$

Differentiating $w_n(t)$ defined in (19) and using (18) and Green's first identity, we have

$$\begin{aligned}&\mathrm{d}w_n(t) = [(-\lambda_n + q)w_n(t) - \mathbf{b}_n^{\mathrm{T}}\mathbf{v}(t_j) + f_n(t)]\mathrm{d}t \\ &\qquad +g_n(t)\mathrm{d}\mathcal{W}(t), \quad t \in [t_j, t_{j+1}), \\ &w_n(0) = \langle w(\cdot, 0), \phi_n\rangle_{\mathcal{O}},\end{aligned} \tag{20}$$

where

$$\begin{aligned}&\mathbf{b}_n = [\langle\psi_1, \phi_n\rangle_{\mathcal{O}}, \cdots, \langle\psi_N, \phi_n\rangle_{\mathcal{O}}]^{\mathrm{T}} \\ &\quad \overset{(11)}{=} [\tfrac{-b_1}{\lambda_n - \mu_1}\langle\tfrac{\partial\phi_1}{\partial\mathbf{n}}, \tfrac{\partial\phi_n}{\partial\mathbf{n}}\rangle_{\Gamma_2}, \cdots, \tfrac{-b_N}{\lambda_n - \mu_N}\langle\tfrac{\partial\phi_N}{\partial\mathbf{n}}, \tfrac{\partial\phi_n}{\partial\mathbf{n}}\rangle_{\Gamma_2}]^{\mathrm{T}}, \\ &f_n(t) = \langle f(w(\cdot, t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)), \phi_n\rangle_{\mathcal{O}}, \\ &g_n(t) = \langle g(w(\cdot, t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)), \phi_n\rangle_{\mathcal{O}}.\end{aligned}$$

Introduce the notations:

$$\begin{aligned}&A_0 = \mathrm{diag}\{-\lambda_n + q\}_{n=1}^N, \quad \mathbf{B}_0 = [\mathbf{b}_1, \ldots, \mathbf{b}_N]^{\mathrm{T}}, \\ &\tilde{A} = \mathrm{diag}\{\Xi_0, A_0\}, \qquad\quad \tilde{\mathbf{B}} = [I_N, -\mathbf{B}_0^{\mathrm{T}}]^{\mathrm{T}}.\end{aligned} \tag{21}$$

By using Assumption 1 and Lemmas 7.1 and 7.2 in [21], we obtain that the pair $(A_0, \mathbf{B}_0)$ is controllable, which implies that $(\tilde{A}, \tilde{\mathbf{B}})$ is controllable. Let $K \in \mathbb{R}^{N\times 2N}$ be the controller gain (it will be found from LMIs (25) below). According to the network-based control described in Fig. 1, we propose a dynamic controller defined by (17) and

$$\begin{aligned}&\mathbf{v}(t_j) = -KX(t_j - \eta_j), \ t \in [t_j, t_{j+1}), \\ &X(t) = \mathrm{col}\{\mathbf{u}(t), w_1(t), \ldots, w_N(t)\},\end{aligned} \tag{22}$$

which is calculated on the controller side.

D. Well-posedness

Consider the state $\xi(t) = \mathrm{col}\{\mathbf{u}(t), w(\cdot, t)\}$. Then system (18) subject to the control input (22) can be presented as

$$\mathrm{d}\xi(t) = [\mathcal{A}_1\xi(t) + F_1(\xi(t)) + F_2(t)]\mathrm{d}t + G(\xi(t))\mathrm{d}\mathcal{W}(t), \tag{23}$$

with $\mathcal{A}_1 = \mathrm{diag}\{\Xi_0, -\mathcal{A}\}$ where $\mathcal{A}$ is given by (3) and Ξ_0 is defined below (15), and

$$\begin{aligned}&F_1(\xi(t)) = \begin{bmatrix} 0 \\ qw(\cdot, t) + f(w(\cdot, t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)) \end{bmatrix}, \\ &F_2(t) = \begin{bmatrix} -I \\ \psi^{\mathrm{T}}(\cdot) \end{bmatrix} K \begin{bmatrix} \mathbf{u}(t - \tau(t)) \\ \{\langle\phi_n, w(\cdot, t - \tau(t))\rangle_{\mathcal{O}}\}_{n=1}^N \end{bmatrix}, \\ &G(\xi(t)) = \begin{bmatrix} 0_{N\times 1} \\ g(w(\cdot, t) + \psi^{\mathrm{T}}(\cdot)\mathbf{u}(t)) \end{bmatrix}.\end{aligned}$$

Let $\mathcal{H} = \mathbb{R}^N \times L^2(\mathcal{O})$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}^2 = |\cdot|^2 + \|\cdot\|_{L^2(\mathcal{O})}^2$. Take $\mathcal{V} = \mathbb{R}^N \times H^1(\mathcal{O})$ with norm $\|\cdot\|_{\mathcal{V}}^2 = |\cdot|^2 + \|\cdot\|_{H^1(\mathcal{O})}^2$, and $\mathcal{V}' = \mathbb{R}^N \times H_0^{-1}(0, 1)$. We see that $\mathcal{A}_1$ satisfies conditions B.1-B.3 on page 198 in [32]. From (2), it can be verified for any $\xi_1, \xi_2 \in \mathcal{H}$,

$$\begin{aligned}&\|F_1(\xi)\|_{\mathcal{H}}^2 + \|G(\xi)\|_{\mathcal{H}}^2 \le 2(\rho^2 + 1)\max\{\sigma_f^2, \sigma_g^2\}\|\xi\|_{\mathcal{H}}^2, \\ &\|F_1(\xi_1) - F_1(\xi_2)\|_{\mathcal{H}}^2 + \|G(\xi_1) - G(\xi_2)\|_{\mathcal{H}}^2 \\ &\le 2(\rho^2 + 1)\max\{\sigma_f^2, \sigma_g^2\}\|\xi_1 - \xi_2\|_{\mathcal{H}}^2.\end{aligned}$$

We first consider $t \in [0, t_0]$ (assume $t_0 > 0$, otherwise we first consider $t \in [0, t_1]$). Since $F_2(t)$ depends on z_0 only for $t \in [0, t_0]$, we have $\mathbb{E}\int_0^{t_0}[\|F_2(s)\|_{\mathcal{H}}^2 < M_0$ for some constant $M_0 > 0$. Then by [32, Theorem 6.7.4], for any initial value $\xi_0 \in L^2(\Omega; \mathcal{H})$ and $\xi_0 \in \mathcal{D}(\mathcal{A}_1)$ almost surely, the closed loop system (23) has a unique strong solution satisfying $\xi \in L^2(\Omega; C([0, t_0]; \mathcal{H})) \cap L^2([0, t_0] \times \Omega; \mathcal{V})$, such that $\xi(t) \in \mathcal{D}(\mathcal{A}_1)$ almost surely and is adapted to $\mathcal{F}_t$ for $t \in [0, t_0]$. Then by the step method on $\{[t_k, t_{k+1}]\}_{k=1}^\infty$ with initial conditions $\xi(t_k) \in \mathcal{D}(\mathcal{A})$, we obtain, for $\xi_0 \in \mathcal{D}(\mathcal{A}_1)$ almost surely, existence of a unique solution $\xi \in L^2(\Omega; C([0, \infty); \mathcal{H})) \cap L^2([0, \infty)\backslash\mathcal{J} \times \Omega; \mathcal{V})$, where $\mathcal{J} = \{t_j\}_{j=0}^\infty$, such that $\xi(t) \in \mathcal{D}(\mathcal{A}_1)$ almost surely.

III. Mean-square L^2 exponential stabilization

By (16), (17), (20), and (22), we obtain the closed-loop system for $t \ge 0$:

$$\mathrm{d}X(t) = \mathbf{F}(t)\mathrm{d}t + \mathbf{G}(t)\mathrm{d}\mathcal{W}(t), \ t \ge 0, \tag{24a}$$

$$\begin{aligned}\mathrm{d}w_n(t) = [&(-\lambda_n + q)w_n(t) + \mathbf{b}_n^{\mathrm{T}}K(X(t) - \nu(t)) \\ &+ f_n(t)]\mathrm{d}t + g_n(t)\mathrm{d}\mathcal{W}(t), \quad n > N,\end{aligned} \tag{24b}$$

where

$$\begin{aligned}&\mathbf{F}(t) = (\tilde{A} - \tilde{\mathbf{B}}K)X(t) + \tilde{\mathbf{B}}K\nu(t) + F^N(t), \\ &\nu(t) = X(t) - X(t - \tau), \ \text{with } \tau(t) \text{ defined in (16)}, \\ &F^N(t) = \mathrm{col}\{0_{N\times 1}, f_1(t), \ldots, f_N(t)\}, \\ &\mathbf{G}(t) = G^N(t), \ G^N(t) = \mathrm{col}\{0_{N\times 1}, g_1(t), \ldots, g_N(t)\}.\end{aligned}$$

Next, we derive LMI conditions for finding controller dimension N and gain K, upper bounds on Lipschitz constants and sampling intervals, ensuring the global mean-squre L^2 exponential stability of system (24). The main result is given in the following theorem:

Theorem 1: Consider system (17)-(18) with f, g satisfying (2) for some $\sigma_f, \sigma_g > 0$, and the control law (22). Assume $z_0 \in \mathcal{D}(\mathcal{A})$ almost surely and $z_0 \in L^2(\Omega; L^2(\mathcal{O}))$. Let $\delta > 0$ be a desired decay rate, $N \in \mathbb{N}$ satisfy (5) and Assumption 1 hold. Let there exist scalars $\alpha_1, \alpha_2, \alpha_3 > 0$, $0 < \beta_1 < 1$, matrices $0 < \bar{P}, \bar{S}, \bar{R}, \bar{Q} \in \mathbb{R}^{2N\times 2N}$, $\bar{M}, \bar{P}_2 \in \mathbb{R}^{2N\times 2N}$, $Y \in \mathbb{R}^{N\times 2N}$ and tuning parameters $\rho, \varepsilon_1, \varepsilon_2 > 0$ such that the following LMIs are feasible

$$\begin{bmatrix} \bar{R} & \bar{M} \\ * & \bar{R} \end{bmatrix} \ge 0, \tag{25a}$$

$$\begin{bmatrix} -\lambda_{N+1} + q + \delta + \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) & \sigma_f & \sigma_g \\ * & -\alpha_3 & 0 \\ * & * & -\beta_1 \end{bmatrix} < 0, \tag{25b}$$

$$\begin{bmatrix} -\varepsilon_2(\bar{P}_2 + \bar{P}_2^{\mathrm{T}}) + \bar{P} + \tau_M\bar{Q} & \varepsilon_2\beta_1 I \\ * & -\beta_1 I \end{bmatrix} < 0, \tag{25c}$$

$$\left[\begin{array}{ccc|c|cc} \hat{\Theta}_{11} & \hat{\Theta}_{12} & \alpha_3 I & & & \\ * & \hat{\Theta}_{22} & \varepsilon_1\alpha_3 I & \hat{\Theta}_{14} & \hat{\Theta}_{15} & 0 \\ * & * & -\alpha_3 I & & & \\ \hline & * & & \hat{\Theta}_{44} & 0 & \hat{\Theta}_{46} \\ \hline & * & & * & \hat{\Theta}_{55} & 0 \\ & * & & * & * & -\frac{\alpha_2}{\|\psi\|_N^2}I \end{array}\right] < 0, \tag{25d}$$

where

$$\begin{aligned}&\hat{\Theta}_{11} = \bar{P}_2^{\mathrm{T}}\tilde{A}^{\mathrm{T}} + \tilde{A}\bar{P}_2 - Y^{\mathrm{T}}\tilde{\mathbf{B}}^{\mathrm{T}} - \tilde{\mathbf{B}}Y + (1 - \varepsilon_\tau)\bar{S} + 2\delta\bar{P}, \\ &\hat{\Theta}_{12} = \bar{P} - \bar{P}_2 + \varepsilon_1\bar{P}_2^{\mathrm{T}}\tilde{A}^{\mathrm{T}} - \varepsilon_1 Y^{\mathrm{T}}\tilde{\mathbf{B}}^{\mathrm{T}}, \\ &\hat{\Theta}_{22} = -\varepsilon_1(\bar{P}_2 + \bar{P}_2^{\mathrm{T}}) + \tau_M^2\bar{R}, \ \hat{\Theta}_{46} = \mathrm{col}\{Y^{\mathrm{T}}, 0, 0, 0\}, \\ &\hat{\Theta}_{14} = \begin{bmatrix} \varepsilon_\tau\bar{S} + \tilde{\mathbf{B}}Y & \varepsilon_\tau\bar{S} & 0 & 0 \\ \varepsilon_1\tilde{\mathbf{B}}Y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\Theta}_{15} = \begin{bmatrix} \sigma_g\bar{P}_2^{\mathrm{T}}\Lambda_2^{\frac{1}{2}} & \bar{P}_2^{\mathrm{T}}\Lambda_2^{\frac{1}{2}} & Y^{\mathrm{T}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &\hat{\Theta}_4 = \begin{bmatrix} -\varepsilon_\tau(\bar{S} + \bar{R}) & -\varepsilon_\tau(\bar{M} + \bar{S}) & \varepsilon_\tau\bar{R} & \varepsilon_\tau\bar{M} \\ * & -\varepsilon_\tau(\bar{R} + \bar{S}) & \varepsilon_\tau\bar{M}^{\mathrm{T}} & \varepsilon_\tau\bar{R} \\ * & * & -\varepsilon_\tau(\bar{R} + \bar{Q}) & 0 \\ * & * & * & -\varepsilon_\tau(\bar{R} + \bar{Q}) \end{bmatrix}, \\ &\hat{\Theta}_{55} = \mathrm{diag}\{-\tfrac{\beta_1}{2}I, -\tfrac{\alpha_3}{2\sigma_f^2}I, -\tfrac{\alpha_1}{\|\psi\|_N^2}I\}, \quad \varepsilon_\tau = \mathrm{e}^{-2\delta\tau_M}.\end{aligned}$$

Then solution $\mathbf{u}(t)$, $w(x, t)$ to (17), (18) under the control law (22) with controller gain $K = Y\bar{P}_2^{-1}$ satisfy

$$\mathbb{E}[|\mathbf{u}(t)|^2 + \|w(\cdot, t)\|_{L^2(\mathcal{O})}^2] \le D_0\mathrm{e}^{-2\delta t}\mathbb{E}\|z_0\|_{L^2(\mathcal{O})}^2, \tag{26}$$

for some $D_0 > 0$ and $t \geq 0$.

Proof: We consider the Lyapunov functional

$$\begin{aligned} &V(t) = V_P(t) + V_S(t) + V_R(t) + V_Q(t) + V_{\rm tail}(t), \\ &V_P(t) = |X(t)|_P^2, \quad V_{\rm tail}(t) = \textstyle\sum_{n=N+1}^{\infty} w_n^2(t), \\ &V_S(t) = \textstyle\int_{t-\tau_M}^{t} {\rm e}^{-2\delta(t-s)}|X(s)|_S^2{\rm d}s, \\ &V_R(t) = \tau_M \textstyle\int_{-\tau_M}^{0}\int_{t+\theta}^{t} {\rm e}^{-2\delta(t-s)}|\mathbf{F}(s)|_R{\rm d}s{\rm d}\theta, \\ &V_Q(t) = \textstyle\int_{-\tau_M}^{0}\int_{t+\theta}^{t} {\rm e}^{-2\delta(t-s)}|\mathbf{G}(s)|_Q{\rm d}s{\rm d}\theta, \end{aligned} \tag{27}$$

where $[P, R, S, Q] = (\bar{P}_2^{-1})^{\rm T}[\bar{P}, \bar{R}, \bar{S}, \bar{Q}]\bar{P}_2^{-1}$ are positive-definite. Without loss of generality we assume $z(\cdot, t) = z_0(\cdot)$ for $t < 0$. In this regard, the solution of system (24) for $t < 0$ is well-defined. The construction of functional $V(t)$ follows [18], which is a stochastic extension of the Lyapunov functional in [13]. The terms V_S and V_P are used to compensate the delay term $\nu(t)$. Here V_S has the same form as the deterministic case, whereas V_R is stochastic extensions of the state-derivative-dependent double integral terms. The term V_Q compensates the stochastic parts of (24a) (see [33]).

Differently from the deterministic case [5], [26], we cannot take generator $\mathcal{L}$ term by term in the infinite sum. Following [17], we use Parseval's equality and present $V_{\rm tail}(t)$ in (27) as

$$\begin{aligned} &V_{\rm tail}(t) = -V_1(t) + V_2(w(t)), V_2(w) = \|w\|_{L^2(\mathcal{O})}^2, \\ &V_1(t) = X^{\rm T}(t)\Lambda_1 X(t), \quad \Lambda_1 = {\rm diag}\{0_{N\times N}, I_N\}. \end{aligned} \tag{28}$$

From the well-posedness in Sec. II-D, we see that $w(t)$ is a strong solution to the following stochastic evolution equation:

$$\begin{aligned} &{\rm d}w(t) = [-\mathcal{A}w(t) + qw(t) + \psi^{\rm T}(\cdot)K[X(t) - \nu(t)] \\ &+f(w(t) + \psi^{\rm T}(\cdot)\mathbf{u}(t))]{\rm d}t + g(w(t) + \psi^{\rm T}(\cdot)\mathbf{u}(t)){\rm d}\mathcal{W}(t). \end{aligned} \tag{29}$$

For $V_2(w)$, calculating the generator $\mathcal{L}$ along (29) for $t \in [t_j, t_{j+1})$ (see [32, P. 228]), we obtain

$$\begin{aligned} &\mathcal{L}V_2(w(t)) \overset{(2)}{\leq} 2\langle w(t), -\mathcal{A}w(t) + qw(t) + \psi^{\rm T}[X(t) - \nu(t)]\rangle_{\mathcal{O}} \\ &\quad +2\langle w(t), f(w(t) + \psi^{\rm T}\mathbf{u}(t))\rangle_{\mathcal{O}} + \sigma_g^2\|w(t) + \psi^{\rm T}\mathbf{u}(t)\|_{L^2(\mathcal{O})}^2 \\ &\leq \textstyle\sum_{n=1}^{\infty} 2(-\lambda_n + q)w_n^2(t) + \sum_{n=1}^{\infty} 2w_n(t)\mathbf{b}_n^{\rm T}K[X(t) - \nu(t)] \\ &\quad + \textstyle\sum_{n=1}^{\infty} 2w_n(t)f_n(t) + 2\sigma_g^2|X(t)|_{\Lambda_2}^2 + 2\sigma_g^2\sum_{n=N+1}^{\infty} w_n^2(t), \end{aligned} \tag{30}$$

where $\Lambda_2 = {\rm diag}\{\Psi^N, I_N\}$ with $\Psi^N = (\langle\psi_i, \psi_j\rangle_{\mathcal{O}})_{i,j=1}^N$ which is a positive semi-definite matrix. Calculating $\mathcal{L}V_1$ along (24a) for $t \in [t_j, t_{j+1})$ (see [34, P. 149]), we have

$$\begin{aligned} &\mathcal{L}V_1(t) + 2\delta V_1(t) = \textstyle\sum_{n=1}^{N} 2(-\lambda_n + q + \delta)w_n^2(t) \\ &\quad + \textstyle\sum_{n=1}^{N} 2w_n(t)\mathbf{b}_n^{\rm T}K[X(t) - \nu(t)] \\ &\quad + \textstyle\sum_{n=1}^{N} 2w_n(t)f_n(t) + |G^N(t)|^2. \end{aligned} \tag{31}$$

From (30) and (31), it follows

$$\begin{aligned} &\mathcal{L}V_{\rm tail}(t) + 2\delta V_{\rm tail}(t) \\ &\leq \textstyle\sum_{n=N+1}^{\infty} 2(-\lambda_n + q + \delta + \sigma_g^2)w_n^2(t) \\ &+ \textstyle\sum_{n=N+1}^{\infty} 2w_n(t)f_n(t) + 2\sigma_g^2|X(t)|_{\Lambda_2}^2 - |G^N(t)|^2 \\ &+ \textstyle\sum_{n=N+1}^{\infty} 2w_n(t)\mathbf{b}_n^{\rm T}K[X(t) - \nu(t)]. \end{aligned} \tag{32}$$

By Young's inequality, we have for $\alpha_1, \alpha_2, \alpha_3 > 0$

$$\begin{aligned} &\textstyle\sum_{n=N+1}^{\infty} 2w_n(t)\mathbf{b}_n^{\rm T}K[X(t) - \nu(t)] \\ &\leq (\alpha_1 + \alpha_2)\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) \\ &\quad + \tfrac{\|\psi\|_N^2}{\alpha_1}|KX(t)|^2 + \tfrac{\|\psi\|_N^2}{\alpha_2}|K\nu(t)|^2 \end{aligned} \tag{33}$$

and

$$\begin{aligned} \textstyle\sum_{n=N+1}^{\infty} 2w_n(t)f_n(t) &\leq \alpha_3\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) \\ &\quad -\tfrac{1}{\alpha_3}|F^N(t)|^2 + \tfrac{1}{\alpha_2}\textstyle\sum_{n=1}^{\infty} f_n^2(t), \end{aligned} \tag{34}$$

where $\|\psi\|_N^2 = \sum_{i=1}^N \|\psi_i\|_N^2$, $\|\psi_i\|_N^2 := \sum_{n=N+1}^{\infty} |\langle\psi_i, \phi_n\rangle_{\mathcal{O}}|^2$. From Parseval's equality, we obtain

$$\begin{aligned} &\textstyle\sum_{n=1}^{\infty} f_n^2(t) = \|f(w(\cdot, t) + \psi^{\rm T}(\cdot)\mathbf{u}(t))\|_{L^2(\mathcal{O})}^2 \\ &\overset{(2)}{\leq} \sigma_f^2\|w(\cdot, t) + \psi^{\rm T}(\cdot)\mathbf{u}(t)\|_{L^2(\mathcal{O})}^2 \\ &\leq 2\sigma_f^2 X^{\rm T}(t)\Lambda_2 X(t) + 2\sigma_f^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t). \end{aligned} \tag{35}$$

Combination of (34) and (35) implies

$$\begin{aligned} &\textstyle\sum_{n=N+1}^{\infty} 2w_n(t)f_n(t) \leq \tfrac{2\sigma_f^2}{\alpha_3}X^{\rm T}(t)\Lambda_2 X(t) \\ &+(\tfrac{2\sigma_f^2}{\alpha_3} + \alpha_3)\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) - \tfrac{1}{\alpha_3}|F^N(t)|^2. \end{aligned} \tag{36}$$

Substitution of (33) and (36) into (32) gives

$$\begin{aligned} &\mathcal{L}V_{\rm tail}(t) + 2\delta V_{\rm tail}(t) \leq 2(\sigma_g^2 + \tfrac{\sigma_f^2}{\alpha_3})X^{\rm T}(t)\Lambda_2 X(t) - |G^N(t)|^2 \\ &+ \textstyle\sum_{n=N+1}^{\infty} 2[-\lambda_n + q + \delta + \sigma_g^2 + \frac{\alpha_1+\alpha_2+\alpha_3}{2} + \frac{\sigma_f^2}{\alpha_3}]w_n^2(t) \\ &-\tfrac{1}{\alpha_3}|F^N(t)|^2 + \tfrac{\|\psi\|_N^2}{\alpha_1}|KX(t)|^2 + \tfrac{\|\psi\|_N^2}{\alpha_2}|K\nu(t)|^2, t \in [t_j, t_{j+1}). \end{aligned} \tag{37}$$

For V_P, V_S, V_R, V_Q, calculating generator $\mathcal{L}$ along stochastic ODE (24a) for $t \in [t_j, t_{j+1})$ (see [34, P. 149]), we have

$$\begin{aligned} &\mathcal{L}V_P(t) + 2\delta V_P(t) = 2X^{\rm T}(t)P\mathbf{F}(t) + |\mathbf{G}(t)|_P^2 + 2\delta|X(t)|_P^2, \\ &\mathcal{L}V_S(t) + 2\delta V_S(t) = |X(t)|_S^2 - \varepsilon_\tau|X(t) - \nu(t) - \theta(t)|_S^2, \\ &\mathcal{L}V_R(t) + 2\delta V_R(t) = \tau_M^2|\mathbf{F}(t)|_R^2 - \varepsilon_\tau\tau_M\textstyle\int_{t-\tau_M}^{t}|\mathbf{F}(s)|_R^2{\rm d}s, \\ &\mathcal{L}V_Q(t) + 2\delta V_Q(t) = \tau_M|\mathbf{G}(t)|_Q^2 - \varepsilon_\tau\textstyle\int_{t-\tau_M}^{t}|\mathbf{G}(s)|_Q^2{\rm d}s, \end{aligned} \tag{38}$$

where $\theta(t) = X(t - \tau) - X(t - \tau_M)$. By employing the Itô isometry (see [35, P. 28]), we have

$$\begin{aligned} &\mathbb{E}[\mathcal{L}V_Q(t) + 2\delta V_Q(t)] = \tau_M\mathbb{E}|G^N(t)|_Q^2 \\ &-\varepsilon_\tau\mathbb{E}|\xi_1(t)|_Q^2 - \varepsilon_\tau\mathbb{E}|\xi_2(t)|_Q^2, \quad t \in [t_j, t_{j+1}), \\ &\xi_1(t) = \textstyle\int_{t-\tau_u}^{t}\mathbf{G}(s){\rm d}\mathcal{W}(s), \xi_2(t) = \int_{t-\tau_M}^{t-\tau_u}\mathbf{G}(s){\rm d}\mathcal{W}(s). \end{aligned} \tag{39}$$

Let $M = (\bar{P}_2^{-1})^{\rm T}\bar{M}\bar{P}_2^{-1} \in \mathbb{R}^{2N\times 2N}$. From (25a), it follows

$$\begin{bmatrix} R & M \\ * & R \end{bmatrix} \geq 0. \tag{40}$$

Applying Jensen's and Park's inequalities (see, e.g., Sec. 3.6.3 of [28]), we obtain

$$\begin{aligned} &\tau_M\textstyle\int_{t-\tau_M}^{t}|\mathbf{F}(s)|_R^2{\rm d}s \\ &\geq \begin{bmatrix} \int_{t-\tau}^{t}F(s){\rm d}s \\ \int_{t-\tau_M}^{t-\tau}F(s){\rm d}s \end{bmatrix}^{\rm T}\begin{bmatrix} R & M \\ * & R \end{bmatrix}\begin{bmatrix} \int_{t-\tau}^{t}F(s){\rm d}s \\ \int_{t-\tau_M}^{t-\tau}F(s){\rm d}s \end{bmatrix} \\ &= \begin{bmatrix} \nu(t) - \xi_1(t) \\ \theta(t) - \xi_2(t) \end{bmatrix}^{\rm T}\begin{bmatrix} R & M \\ * & R \end{bmatrix}\begin{bmatrix} \nu(t) - \xi_1(t) \\ \theta(t) - \xi_2(t) \end{bmatrix}. \end{aligned} \tag{41}$$

Besides, from Parseval's equality and (2) we have

$$\begin{aligned} &|G^N(t)|^2 = \textstyle\sum_{n=1}^{N} g_n^2(t) \leq \sum_{n=1}^{\infty} g_n^2(t) \\ &\leq \sigma_g^2\|w(\cdot, t) + \psi^{\rm T}(\cdot)\mathbf{u}(t)\|_{L^2}^2 \\ &\leq 2\sigma_g^2|X(t)|_{\Lambda_2}^2 + 2\sigma_g^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t). \end{aligned} \tag{42}$$

Let $\eta_1(t) = {\rm col}\{X(t), \mathbf{F}(t), F^N(t), \nu(t), \theta(t), \xi_1(t), \xi_2(t)\}$ and $\eta_2(t) = {\rm col}\{\mathbf{G}(t), G^N(t)\}$. By (37) - (42) and using the descriptor method:

$$\begin{aligned} 0 = 2[X^{\rm T}(t)P_2^{\rm T} &+ \mathbf{F}^{\rm T}(t)P_3^{\rm T}][(\tilde{A} - \tilde{\mathbf{B}}K)X(t) + \tilde{\mathbf{B}}K\nu(t) \\ &+F^N(t) - \mathbf{F}(t)] + 2\mathbf{G}^{\rm T}(t)P_4^{\rm T}[G^N(t) - \mathbf{G}(t)], \end{aligned} \tag{43}$$

where $P_2 = \bar{P}_2^{-1}$, $P_3 = \varepsilon_1 P_2$, $P_4 = \varepsilon_2 P_2$ with tuning parameters $\varepsilon_1, \varepsilon_2 > 0$, we get

$$\begin{aligned} &\mathbb{E}[\mathcal{L}V(t) + 2\delta V(t)] + \beta\mathbb{E}[2\sigma_g^2 X^{\rm T}(t)\Lambda_2 X(t) \\ &\quad +2\sigma_g^2\textstyle\sum_{n=N+1}^{\infty} w_n^2(t) - |G^N(t)|^2] \\ &\leq \mathbb{E}[\eta_1^{\rm T}(t)\Theta\eta_1(t)] + \mathbb{E}[\eta_2^{\rm T}(t)\Xi\eta_2(t)] \\ &\quad +\mathbb{E}[\textstyle\sum_{n=N+1}^{\infty} 2\chi_n z_n^2(t)], \quad t \in [t_j, t_{j+1}), \end{aligned} \tag{44}$$

where $\beta = \frac{1}{\beta_1} - 1 > 0$ and

$$\chi_n = -\lambda_n + q + \delta + (\beta+1)\sigma_g^2 + \sigma_f^2/\alpha_3 + 0.5(\alpha_1+\alpha_2+\alpha_3), \quad n > N, \tag{45a}$$

$$\Xi = \begin{bmatrix} -P_4 - P_4^{\mathrm{T}} + P + \tau_M Q & P_4^{\mathrm{T}} \\ * & -(\beta+1)I \end{bmatrix}, \tag{45b}$$

$$\Theta = \left[\begin{array}{ccc|c} \Theta_{11} & \Theta_{12} & P_2^{\mathrm{T}} & \\ * & \Theta_{22} & P_3^{\mathrm{T}} & \Theta_{14} \\ * & * & -\frac{1}{\alpha_3}I & \\ \hline & * & & \Theta_{44} \end{array}\right]. \tag{45c}$$

The blocks in matrix Θ in (45c) are give by

$$\begin{aligned} \Theta_{11} &= (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}} P_2 + P_2^{\mathrm{T}}(\tilde{A} - \tilde{\mathbf{B}}K) + (1-\varepsilon_\tau)S \\ &\quad + 2\delta P + \tfrac{\|\psi\|_N^2}{\alpha_1} K^{\mathrm{T}}K + 2(\beta+1)\sigma_g^2\Lambda_2 + \tfrac{2\sigma_f^2}{\alpha_3}\Lambda_2, \\ \Theta_{12} &= P - P_2^{\mathrm{T}} + (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}} P_3, \\ \Theta_{22} &= -P_3 - P_3^{\mathrm{T}} + \tau_M^2 R, \\ \Theta_{14} &= \begin{bmatrix} \varepsilon_\tau S + P_2^{\mathrm{T}}\tilde{\mathbf{B}}K & \varepsilon_\tau S & 0 & 0 \\ P_3^{\mathrm{T}}\tilde{\mathbf{B}}K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \Theta_{44} &= \begin{bmatrix} \Theta_{44}^{(1)} & -\varepsilon_\tau(M+S) & \varepsilon_\tau R & \varepsilon_\tau M \\ * & -\varepsilon_\tau(R+S) & \varepsilon_\tau M^{\mathrm{T}} & \varepsilon_\tau R \\ * & * & -\varepsilon_\tau(R+Q) & 0 \\ * & * & * & -\varepsilon_\tau(R+Q) \end{bmatrix}, \\ \Theta_{44}^{(1)} &= \tfrac{\|\psi\|_N^2}{\alpha_2} K^{\mathrm{T}}K - \varepsilon_\tau(S+R). \end{aligned}$$

Multiplying Θ in (45c) by $\mathrm{diag}\{\bar{P}_2, \bar{P}_2, \alpha_3 I, \bar{P}_2, \bar{P}_2, \bar{P}_2, \bar{P}_2\}$ and its transpose, from the right and the left, respectively, using Schur complement, and recalling $P_3 = \varepsilon_1 P_2$, $K = Y\bar{P}_2^{-1}$, we obtain that (25d) implies $\Theta < 0$. Similarly, multiplying Ξ in (45b) by $\mathrm{diag}\{\bar{P}_2, \beta_1\}$ and its transpose, from the right and the left, and recalling $P_4 = \varepsilon_2 P_2$, $\beta = \frac{1}{\beta_1} - 1 > 0$, we obtain that (25c) implies $\Xi < 0$. Since λ_n are non-decreasing, from (45a), it follows that $\chi_n < 0$, $n > N$ iff $\chi_{N+1} < 0$. Using Schur complement and recalling $\beta = \frac{1}{\beta_1} - 1 > 0$, we obtain that (25b) implies $\chi_n < 0$, $n > N$. Then, from (44) we have

$$\mathbb{E}[\mathcal{L}V(t) + 2\delta V(t)] < 0, \quad t \in [t_j, t_{j+1}). \tag{46}$$

Next, we prove that (26) follows from (46). By employing Itô's formula for $e^{2\delta t}V_{\mathrm{finite}}(t)$ and $e^{2\delta t}V_1(t)$, $t \in [t_j, t_{j+1})$, where $V_{\mathrm{finite}}(t) = V_P(t) + V_S(t) + V_R(t) + V_Q(t)$, along stochastic ODE (24a) (see [34, Theorem 4.18]), and taking expectation on both sides, we have

$$\begin{aligned} &\mathbb{E}[e^{2\delta t}V_{\mathrm{finite}}(t)] = \mathbb{E}[e^{2\delta t_j}V_{\mathrm{finite}}(t_j)] \\ &\quad + \textstyle\int_{t_j}^{t} e^{2\delta s}\mathbb{E}[\mathcal{L}V_{\mathrm{finite}}(s) + 2\delta V_{\mathrm{finite}}(s)]\mathrm{d}s, \\ &\mathbb{E}[e^{2\delta t}V_1(t)] = \mathbb{E}[e^{2\delta t_j}V_1(t_j)] \\ &\quad + \textstyle\int_{t_j}^{t_{j+1}} e^{2\delta s}\mathbb{E}[\mathcal{L}V_1(s) + 2\delta V_1(s)]\mathrm{d}s, \ t \in [t_j, t_{j+1}). \end{aligned} \tag{47}$$

Employing Itô's formula for $e^{2\delta t}V_2(w(t))$, $t \in [t_j, t_{j+1})$, along (29) (see [32, Theorem 7.2.1]) and taking expectation on both sides, we arrive at

$$\begin{aligned} &\mathbb{E}[e^{2\delta t}V_2(w(t))] = \textstyle\int_{t_j}^{t} e^{2\delta s}\mathbb{E}[\mathcal{L}V_2(w(s)) + 2\delta V_2(w(s))]\mathrm{d}s \\ &+ \mathbb{E}[e^{2\delta t_j}V_2(w(t_j))], \quad t \in [t_j, t_{j+1}). \end{aligned} \tag{48}$$

Using (47), (48), and the definition $V(t) = V_{\mathrm{finite}}(t) - V_1(t) + V_2(w(t))$ (see (27) and (28)), we obtain

$$\begin{aligned} e^{2\delta t}\mathbb{E}V(t) &= e^{2\delta t_j}\mathbb{E}V(t_j) + \textstyle\int_{t_j}^{t} e^{2\delta s}\mathbb{E}[\mathcal{L}V(s) + 2\delta V(s)]\mathrm{d}s \\ &\overset{(46)}{\le} e^{2\delta t_j}\mathbb{E}V(t_j), \ t \in [t_j, t_{j+1}), \end{aligned}$$

which implies $\mathbb{E}V(t) \le e^{-2\delta(t-t_j)}\mathbb{E}V(t_j)$, $t \in [t_j, t_{j+1})$. From Sec. II-D, it follows that $\mathbb{E}|X(t)|^2$ and $\mathbb{E}\|w(\cdot,t)\|_{L^2(\mathcal{O})}^2$ are continuous for all $t \ge 0$. This combined with the definition of $V(t)$ in (27) and (28) implies that $\mathbb{E}V(t)$ is continuous, i.e., $\mathbb{E}V(t_j^-) = \mathbb{E}V(t_j)$. Then we obtain $\mathbb{E}V(t) \le e^{-2\delta t}\mathbb{E}V(0)$ for all $t > 0$, which implies (26). ∎

Remark 6: The suggested stochastic descriptor method (i.e., (43)) is a stochastic extension of the deterministic case in [28, Sec. 5], leading to σ_g (Lipschitz constant for the multiplicative noise term g) dependent LMIs (25). Here we introduce an additional free matrix P_4 in (43) for the controller gain design, otherwise, (45b) becomes $P + \tau_M Q - (\beta+1)I < 0$. It is equivalent to $\bar{P} + \tau_M\bar{Q} - (\beta+1)\bar{P}_2^{\mathrm{T}}\bar{P}_2 < 0$, which is nonlinear.

Remark 7: In Theorem 1, LMIs (25) are provided for finding minimal N and as large as possible upper bounds on $\sigma_f, \sigma_g, \tau_M$. For example, to find the upper bound on τ_M, we take τ_M as a tuning parameter, starting from a small value and progressively enlarging it till the LMIs become infeasible. We theoretically prove that inequalities (25) (i.e., (40), (45)) are always feasible for large enough N and small enough $\tau_M, \sigma_f, \sigma_g > 0$. We choose $\alpha_1 = \alpha_2 = \frac{\lambda_{N+1}}{2} > 0$, $\alpha_3 \to 0^+$ and $\rho = 1$. From $\|\psi\|_N^2$ defined below (34) and the fact that $\|\psi_i\|_{L^2}^2 = \rho = 1$, we obtain $\|\psi\|_N^2 \le N$. From Lemma 1, we have $\frac{1}{\alpha_1}\|\psi\|_N^2 = \frac{1}{\alpha_2}\|\psi\|_N^2 < \frac{2N}{\lambda_{N+1}} \to \frac{|\mathcal{O}|}{2\pi}$, $N \to \infty$. Therefore, there exists N-independent $\chi_0 > 0$ such that $\frac{1}{\alpha}\|\psi\|_N^2 \le \chi_0$ for all N. We take $\tau_M, \sigma_f, \sigma_g \to 0^+$, $\rho = 1$, $S = 0$, $M = 0$, $Q = R$, $P_2 = P_4 = P$, $P_3 = P_3^{\mathrm{T}}$. Since $\{\lambda_n\}_{n=1}^{\infty}$ are non-decreasing, it is clear that (40) and (45a) hold for large N. By using Schur complement, we find that (45b) and (45c) are feasible if

$$-(\beta+1)I + P < 0, \tag{49a}$$

$$\begin{bmatrix} \Theta_0 & (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}}P_3 & P\tilde{\mathbf{B}}K \\ * & -2P_3 & P_3\tilde{\mathbf{B}}K \\ * & * & \chi_0 K^{\mathrm{T}}K - \frac{1}{2}R \end{bmatrix} < 0, \tag{49b}$$

$$\Theta_0 = (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}}P + P^{\mathrm{T}}(\tilde{A} - \tilde{\mathbf{B}}K) + 2\delta P + \chi_0 K^{\mathrm{T}}K.$$

Letting $R = rI$ and $P_3 = p_3 I$ with $r \to \infty$ and $p_3 \to 0^+$, we find that (49b) holds iff $\Theta_0 < 0$. We fix N_0 such that

$$-\lambda_n + q + \delta + \tfrac{3}{2}\sigma_f + \sigma_g^2 < 0, \ -\mu_n + q < 0, \quad n > N_0.$$

Design $K \in \mathbb{R}^{N\times 2N}$ and $P \in \mathbb{R}^{2N\times 2N}$ be of the form

$$K = \left[\begin{array}{cc|cc} K_u & 0 & K_w & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], P = \left[\begin{array}{cc|cc} P_{11} & 0 & P_{12} & 0 \\ * & p_u I & 0 & 0 \\ \hline * & * & P_{22} & 0 \\ * & * & * & p_w I \end{array}\right],$$

where $p_u, p_w > 0$, $K_u, K_w \in \mathbb{R}^{N_0\times N_0}$ and $0 < P_0 = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} \in \mathbb{R}^{2N_0\times 2N_0}$ are to be determined later. Let

$$\hat{\mathbf{B}}_0 = \left[\begin{array}{c} I_{N_0} \\ \hline -\begin{bmatrix} \langle\psi_1,\phi_1\rangle_{\mathcal{O}} & \cdots & \langle\psi_{N_0},\phi_1\rangle_{\mathcal{O}} \\ \vdots & \ddots & \vdots \\ \langle\psi_1,\phi_{N_0}\rangle_{\mathcal{O}} & \cdots & \langle\psi_{N_0},\phi_{N_0}\rangle_{\mathcal{O}} \end{bmatrix} \end{array}\right], \ K_0 = [K_u, K_w],$$

$$\hat{A}_0 = \mathrm{diag}\{-\mu_1, \ldots, -\mu_{N_0}, -\lambda_1, \ldots, -\lambda_{N_0}\} + qI_{2N_0}.$$

By using Schur complement and choosing $p_u, p_w = \frac{1}{N}$, we find that $\Theta < 0$ for $N \to \infty$ if

$$\begin{aligned} &P_0(\hat{A}_0 - \hat{\mathbf{B}}_0K_0) + (\hat{A}_0 - \hat{\mathbf{B}}_0K_0)^{\mathrm{T}}P_0 + 2\delta P_0 \\ &+ \chi_0 K_0^{\mathrm{T}}K_0 < 0. \end{aligned} \tag{50}$$

Since $(\tilde{A}, \tilde{\mathbf{B}}_0)$ is stabilizable, $(\hat{A}_0, \hat{\mathbf{B}}_0)$ is also stabilizable. We can choose $K_0 = [K_u, K_w] \in \mathbb{R}^{N_0 \times 2N_0}$ such that $\hat{A}_0 - \hat{B}_0 K_0 + \delta I$ is Hurwitz. Let $P_0 \in \mathbb{R}^{2N_0 \times 2N_0}$ be such that

$$P_0(\hat{A}_0 - \hat{B}_0 K_0 + \delta I) + (\hat{A}_0 - \hat{B}_0 K_0 + \delta I)^{\mathrm{T}} P_0 = -\chi I, \tag{51}$$

where $\chi > 0$ is independent of N and satisfies $\chi I > \chi_0 K_0^{\mathrm{T}} K_0$. Then $P_0 = O(1)$, $N \to \infty$. Substituting (51) into (50), we find that (50) is feasible. Since we take $p_u, p_w = \frac{1}{N} < 1$, we obtain $P = O(1)$, $N \to \infty$. Take $\beta = N$. It is obvious that (49) hold true for $N \to \infty$. By continuity, (40) and (45) are feasible for small enough $\tau_M, \sigma_f, \sigma_g = \frac{1}{N^2}$ and large enough $N \to \infty$.

***Remark** 8:* We can employ the controller dimension N and the gain K obtained from the deterministic system design (see [18, Sec. 3.1]) or continuous-time controller design (see [26, Remark 2.2]), but this leads to more conservative results (see Tables II and III in Sec. IV). For the continuous-time controller design, we use LMIs (42) - (44) in the conference version [29] to get gain K, and by (37) - (41), we obtain (26) provided

$$\begin{bmatrix} -\lambda_{N+1} + q + \delta + \alpha_3 \sigma_f^2 + (1+\beta)\sigma_g^2 & 1 & 1 & 1 \\ * & -2\alpha_1 & 0 & \\ * & * & -2\alpha_2 & 0 \\ * & * & * & -2\alpha_3 \end{bmatrix} < 0,$$
$$P + \tau_M Q - (1+\beta) I < 0,$$
$$\left[\begin{array}{cc|cccc} \Theta_{11} & P & \varepsilon_\tau S + P\tilde{\mathbf{B}}K & \varepsilon_\tau S & 0 & 0 \\ * & -\alpha_3 I & 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{4}{c}{\Theta_{33}} \end{array}\right]$$
$$+\tau_M^2 \Lambda_3^{\mathrm{T}} R \Lambda_3 < 0, \qquad \Lambda_3 = [\tilde{A} - \tilde{\mathbf{B}}K, I_N, \tilde{\mathbf{B}}K, 0, 0, 0],$$
$$\Theta_{11} = (\tilde{A} - \tilde{\mathbf{B}}K)^{\mathrm{T}} P + P(\tilde{A} - \tilde{\mathbf{B}}K) + (1 - \varepsilon_\tau) S$$
$$+2\delta P + \alpha_1 \|\psi\|_N^2 K^{\mathrm{T}} K + 2(\beta+1)\sigma_g^2 \Lambda_2 + 2\alpha_3 \sigma_f^2 \Lambda_2,$$
$$\Theta_{33} = \begin{bmatrix} \Theta_{33}^{(1)} & -\varepsilon_\tau(M+S) & \varepsilon_\tau R & \varepsilon_\tau M \\ * & -\varepsilon_\tau(R+S) & \varepsilon_\tau M^{\mathrm{T}} & \varepsilon_\tau R \\ * & * & -\varepsilon_\tau(R+Q) & 0 \\ * & * & * & -\varepsilon_\tau(R+Q) \end{bmatrix},$$
$$\Theta_{33}^{(1)} = \alpha_2 \|\psi\|_N^2 K^{\mathrm{T}} K - \varepsilon_\tau(S+R). \tag{52}$$

IV. A NUMERICAL EXAMPLE

In this section, we consider a rectangular domain $\mathcal{O} = (0, a_1) \times (0, a_2)$ with $a_1 = a_2 = 1$ and boundary (6). In this case, λ_n, ϕ_n are given by (7), (8), and ψ_i are given by (12). We consider system (1) where f, g satisfy (2). Consider $q = 20$ and $q = 49.4$, respectively, which results in an unstable open-loop system with 1 unstable mode for $q = 20$ and 3 unstable modes for $q = 49.4$, respectively. We will consider $\rho = 0.05$ and $\rho = \frac{1}{\sqrt{2}}$, respectively. Here ρ is the tuning parameter introduced below (10) satisfying $\|\psi_i\|_{L^2(\mathcal{O})} = \rho$. Note that $\rho = 1/\sqrt{2}$ corresponds [8], [9] for 1D case.

First, we choose $\delta = 10^{-4}$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 1$, and fix $\tau_M = 0.01$ for $q = 20$ and $\tau_M = 0.001$ for $q = 49.4$. The feasibility of LMIs (25d), (25a) and (25b) was verified to obtain $\sigma_f^{\max}$ (the maximal value of σ_f) for $\sigma_g = 0.1$ and $\sigma_g^{\max}$ (the maximal value of σ_g) for $\sigma_f = 0.1$. The results are given in Table I. From Table I, we see that the choice $\rho = 0.05$ leads to larger $\sigma_f^{\max}$ and $\sigma_g^{\max}$ than $\rho = \frac{1}{\sqrt{2}}$. In particular, for $N = 5$ and $\sigma_g = 0.1$, the value of $\sigma_f^{\max}$ increases by 23.85% for $q = 20$ and 59.09% for $q = 49.4$.

We next fix $\sigma_f = 2$, $\sigma_g = 0.1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 10$ for $q = 20$ and $\sigma_f = 0.5$, $\sigma_g = 0.1$, $\varepsilon_1 = 0.02$, $\varepsilon_2 = 4$ for $q = 49.4$. The LMIs (25d)-(25b) (descriptor method) and LMIs (52) with the gain K obtained from continuous-time design (by solving LMIs (42)-(44) in [29]) were verified, respectively, for $N \in \{1, \ldots, 5\}$ ($q = 20$) and $N \in \{4, \ldots, 8\}$ ($q = 49.4$) to obtain $\tau_{\max}$ (the maximal value of τ_M) which preserves the feasibility. The results are given in Table II ($q = 20$) and Table III ($q = 49.4$), respectively. From Tables II and III, we see that the controller design via descriptor method allows larger sampling intervals and transmission delays than the design based on the non-delayed case as in [26]. In particular, for $\rho = 0.05$ and $N = 5$, $\tau_{\max}$ increases by 131% for $q = 20$ and by 116% for $q = 49.4$. Moreover, we see that the choice $\rho = 0.05$ leads to larger sampling intervals than $\rho = \frac{1}{\sqrt{2}}$. It should be noted that for the gain K obtained from the deterministic system design as in [18], for both cases, the LMIs (52) are not feasible even for $\tau_{\max} = 0$.

TABLE I
$\sigma_f^{\max}$ AND $\sigma_g^{\max}$ FOR $N = \{1, \ldots, 5\}$ WITH $q = 20$ ($\varepsilon_1 = 0.01, \varepsilon_2 = 1, \tau_M = 0.01$) OR $q = 49.4\}$ ($\varepsilon_1 = 0.01, \varepsilon_2 = 2, \tau_M = 0.001$).

	$\sigma_f^{\max}$ for $\sigma_g = 0.1$				$\sigma_g^{\max}$ for $\sigma_f = 0.1$			
q	20		49.4		2 |0		49.4	
$N\backslash\rho$	0.05	$\frac{1}{\sqrt{2}}$	0.05	$\frac{1}{\sqrt{2}}$	0.05	$\frac{1}{\sqrt{2}}$	0.05	$\frac{1}{\sqrt{2}}$
1	2.61	2.14	–	–	1.18	0 |.88	–	–
2	2.52	2.05	–	–	1.16	0.85	–	–
3	5.04	4.08	0.51	0.27	1.96	1 |.52	0.19	0.12
4	5.14	4.15	0.82	0.53	1.99	1 |.55	0.28	0.15
5	5.08	4.09	0.70	0.44	1.96	1 |.53	0.21	0.13

TABLE II
$\tau_{\max}$ FOR $q = 20$, $\sigma_f = 2$, $\sigma_g = 0.1$, AND $N \in \{1, \ldots, 5\}$: THEOREM 1 (DESCRIPTOR METHOD WITH $\varepsilon_1 = 0.1, \varepsilon_2 = 10$) VS REMARK 8 (CONTINUOUS-TIME DESIGN).

N	$\rho = 0.05$		$\rho = 1/\sqrt{2}$	
	Theorem 1	Remark 8	Theorem 1	Remark 8
1	0.0291	0.0210	0.0097	0.0065
2	0.0280	0.0191	0.0072	0.0047
3	0.0807	0.0384	0.0487	0.0273
4	0.0807	0.0373	0.0492	0.0270
5	0.0807	0.0348	0.0488	0.0263

TABLE III
$\tau_{\max}$ FOR $q = 49.4$, $\sigma_f = 0.5$, $\sigma_g = 0.1$, AND $N \in \{4, \ldots, 8\}$: THEOREM 1 (DESCRIPTOR METHOD WITH $\varepsilon_1 = 0.02, \varepsilon_2 = 4$) VS REMARK 8 (CONTINUOUS- TIME DESIGN).

N	$\rho = 0.05$		$\rho = 1/\sqrt{2}$	
	Theorem 1	Remark 8	Theorem 1	Remark 8
4	0.0058	0.0030	0.0040	0.0016
5	0.0054	0.0025	0.0033	0.0011
6	0.0090	0.0051	0.0074	0.0039
7	0.0089	0.0052	0.0074	0.0036
8	0.0092	0.0056	0.0081	0.0039

For simulations of system (17), (18) with control (22), inspired by the parabolic sine-Gordon model [36], we consider $f(z) = \sigma_f \sin z$. Assume that σ_f undergoes random perturbations and is replaced by $\sigma_f \to \sigma_f + \sigma_g \dot{\mathcal{W}}(t)$ (see [37]) leading to the stochastic term $\sigma_g \sin z \mathrm{d}\mathcal{W}(t)$. We consider $q = 49.4$, $\rho = 0.05$, $\sigma_f = 0.5$, and $\sigma_g = 0.1$. Fix $N = 4$. Table III shows that the upper bound of τ_M is 0.0059. We get the controller gain from LMIs (25):

$$K = \begin{bmatrix} 320.7 & 0 & -117.4 & 0 & 5223.1 & 0 & -138.4 & 0 \\ 0 & 63.3 & 0 & -3.7 & 0 & 81.9 & 0 & -52.1 \\ 357.7 & 0 & -129.0 & 0 & 7988.4 & 0 & -337.9 & 0 \\ 0 & 35.0 & 0 & 16.9 & 0 & 822.7 & 0 & -174.1 \end{bmatrix}.$$

We take $h = 0.004$ and $\eta_M = 0.0018$ (clearly, $\tau_M = h + \eta_M = 0.0058$). The variable sampling instances on the controller side were generated by $s_{j+1} = s_j + 0.5(1 + U_{1,j})h$, $\eta_j = U_{2,j} \cdot \eta_M$, where $U_{1,j} \sim Unif(0,1)$, $U_{2,j} \sim Unif(0,1)$. Take the initial condition $z(x,0) = 10x_1(x_1 - 1)\sin(\pi x_2)$. The simulation was carried out by using the FTCS (Forward Time Centered Space) method and the Euler-Maruyama method with time step 0.0001 and space step 0.05. The simulation results are presented in Fig. 2 and confirm the theoretical analysis. The stability of the closed-loop systems in simulations was preserved for larger $h = 0.018$ (compared to theoretical value $h = 0.004$), which may indicate that our approach is somewhat conservative in this example.

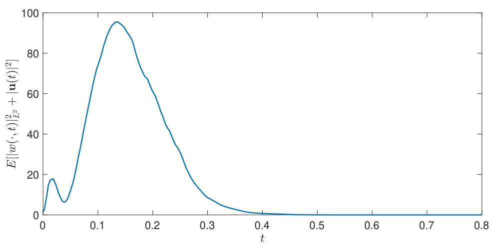

Fig. 2. $\mathbb{E}[|w(\cdot,t)|^2 + |\mathbf{u}(t)|^2]$ ($\mathbb{E}$ means taking average over 50 sample trajectories).

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